RELAXED EULER SYSTEMS AND CONVERGENCE TO NAVIER-STOKES EQUATIONS

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Abstract. We consider the approximation of Navier-Stokes equations for a Newtonian fluid by Euler type systems with relaxation both in compressible and incompressible cases. This requires to decompose the second-order derivative terms of the velocity into first-order ones. Usual decompositions lead to approximate systems with tensor variables. We construct approximate systems with vector variables by using Hurwitz-Radon matrices. These systems are written in the form of balance laws and admit strictly convex entropies, so that they are symmetrizable hyperbolic. For smooth solutions, we prove the convergence of the approximate systems to the Navier-Stokes equations in uniform time intervals. Global-in-time convergence is also shown for the initial data near constant equilibrium states of the systems. These convergence results are established not only for the approximate systems with vector variables but also for those with tensor variables.

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1. Introduction

Euler and Navier-Stokes equations are two fundamental models in fluid mechanics. There is a huge number of studies on mathematical analysis around these equations. We refer to [22, 21, 23, 9, 24] and references therein for mathematical results. It is well known that Euler equations can be derived from Navier-Stokes equations as viscosity coefficients tend to zero. In this paper, we consider the approximation of isentropic Navier-Stokes equations by Euler type equations with relaxation which are referred to as relaxed Euler systems. This approximation problem is studied in both compressible and incompressible cases in whole space $\mathbb{R}^d$ ($d = 1, 2, 3$ in physical situations).

We start with the compressible case. Let $t \geq 0$ be the time variable and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ be the space variable. We denote by $\nu > 0$ the shear viscosity and $\lambda > 0$ the Bulk viscosity. They are supposed to be constants. We consider the isentropic Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \operatorname{div} \pi = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

with the constitutive law for a Newtonian fluid

$$\pi = -\nu\sigma(u) - \lambda(\operatorname{div} u)I_d, \tag{1.2}$$

where

$$\sigma(u) = \nabla u + (\nabla u)^T - \frac{2}{d}(\operatorname{div} u)I_d. \tag{1.3}$$

Here $\rho > 0$ is the density, $u = (u_1, \cdots, u_d)^T \in \mathbb{R}^d$ is the velocity, p is the pressure function and $\nabla u = (\partial_{x_j} u_i)_{1\le i,j\le d}$. We see that π, $\sigma(u)$ and ∇u are tensor variables. In (1.1)-(1.3), I_d is the unit matrix of order d, the symbols T and $\otimes$ stand for the transpose and the tensor product, respectively. Throughout this paper, we assume that p is sufficiently smooth and $p'(\rho) > 0$ for all $\rho > 0$.

For the Navier-stokes equations (1.1)-(1.2), the construction of relaxed Euler systems depends on the way how the term $\operatorname{div}\pi$ of second-order derivatives of u is decomposed into first-order derivative terms. Clearly, there are a lot of ways to do it. Among them a natural one is to replace (1.2) by the Maxwell's constitutive relation [26]

$$\varepsilon\partial_t\pi + \nu\sigma(u) + \lambda(\operatorname{div} u)I_d = -\pi, \tag{1.4}$$

where $\varepsilon > 0$ is a relaxation time. Let us denote by $\operatorname{tr}(\pi)$ the trace of π :

$$\operatorname{tr}(\pi) = \sum_{i=1}^{d} \pi_{ii}, \quad \pi = (\pi_{ij})_{1\le i,j\le d}.$$

Since $\operatorname{tr}(\sigma(u)) = 0$, (1.4) yields

$$\varepsilon\partial_t\operatorname{tr}(\pi) + \operatorname{tr}(\pi) = -\lambda d(\operatorname{div} u),$$

which shows that $\operatorname{tr}(\pi)$ depends on u and in general $\operatorname{tr}(\pi) \neq 0$. Combining (1.1) and (1.4), we obtain a first-order system with relaxation :

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \operatorname{div}\pi = 0, \\ \varepsilon\partial_t\pi + \nu\sigma(u) + \lambda(\operatorname{div} u)I_d = -\pi, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d. \end{cases} \tag{1.5}$$

Formally, as $\varepsilon \to 0$, we recover the Navier-Stokes equations (1.1)-(1.2). To our knowledge, so far system (1.5), in particular on its zero relaxation limit towards (1.1)-(1.2), has not been studied in the literature.

Remark that the tensor π defined in (1.2) is symmetric but the approximate tensor π defined in (1.4) is not always symmetric. By (1.4), the approximate tensor $\pi(t,\cdot)$ is symmetric for all time $t > 0$ if and only if it is symmetric at $t = 0$. Nevertheless, we may consider a slightly more general approximate system by replacing π by $\tilde{\pi} = (\pi + \pi^T)/2$ in the second equation of (1.5). It is easy to see that $\tilde{\pi}$ still satisfies (1.4). Consequently, we obtain an approximate system (1.5) with a symmetric tensor $\tilde{\pi}$ instead of π, even if $\pi(0,\cdot)$ is not symmetric. Thus, the symmetry of π is not a restriction condition. For this reason, throughout this paper, we suppose that the tensor π defined in (1.4) is symmetric.

In a recent paper [39] for $d = 3$, by splitting $\nu\sigma(u) + \lambda(\operatorname{div} u)I_d$ into $\nu\sigma(u)$ and $\lambda(\operatorname{div} u)I_d$, the author proposed a similar first-order system with relaxation :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \dfrac{1}{\sqrt{\varepsilon_1}} \operatorname{div} \tau_1 + \dfrac{1}{\sqrt{\varepsilon_2}} \nabla \tau_2 = 0, \\ \partial_t \tau_1 + \dfrac{\nu}{\sqrt{\varepsilon_1}} \sigma(u) = -\dfrac{\tau_1}{\varepsilon_1}, \\ \partial_t \tau_2 + \dfrac{\lambda}{\sqrt{\varepsilon_2}} \operatorname{div} u = -\dfrac{\tau_2}{\varepsilon_2}, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \tag{1.6}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are relaxation times, τ_1 is a tensor variable and τ_2 is a scalar variable. The last two equations in (1.6) were called revised Maxwell's constitutive relations in [39]. Comparing to (1.5), system (1.6) admits a special property on the trace of τ_1. Indeed, since $\operatorname{tr}(\sigma(u)) = 0$, $\operatorname{tr}(\tau_1)$ satisfies a linear differential equation of the form

$$\varepsilon_1 \partial_t \operatorname{tr}(\tau_1) + \operatorname{tr}(\tau_1) = 0,$$

which implies that $\operatorname{tr}(\tau_1(t, \cdot)) = 0$ for all $t > 0$ as soon as $\operatorname{tr}(\tau_1(0, \cdot)) = 0$. Under condition $\operatorname{tr}(\tau_1(0, \cdot)) = 0$, the author of [39] built a strictly convex entropy which implies that (1.6) is a symmetrizable hyperbolic system. He also proved that the smooth solution of (1.6) converges to that of (1.1) in Sobolev spaces in a uniform time interval as the relaxation times go to zero. However, the case where $\operatorname{tr}(\tau_1(0, \cdot)) \neq 0$ and the convergence for large time have not been investigated.

In order to see that (1.6) is an approximate system of (1.1) for small parameters ε_1 and ε_2, we introduce

$$\pi_1 = \frac{\tau_1}{\sqrt{\varepsilon_1}}, \quad \pi_2 = \frac{\tau_2}{\sqrt{\varepsilon_2}}.$$

Then (1.6) is rewritten as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \operatorname{div} \pi_1 + \nabla \pi_2 = 0, \\ \varepsilon_1 \partial_t \pi_1 + \nu \sigma(u) = -\pi_1, \\ \varepsilon_2 \partial_t \pi_2 + \lambda \operatorname{div} u = -\pi_2, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d. \end{cases} \tag{1.7}$$

Formally, as $(\varepsilon_1, \varepsilon_2) \to 0$, from the last two equations in (1.7), we have

$$\pi_1 = -\nu\sigma(u), \quad \pi_2 = -\lambda \operatorname{div} u.$$

Substituting these two relations into the first two equations in (1.7), we obtain easily the compressible Navier-Stokes equations.

In what follows, systems (1.5) and (1.7) are referred to as relaxed Euler systems with tensor variables.

The first goal of this paper is to introduce a different approach to construct relaxed Euler systems. This approach is motivated by the theory of symmetrizable hyperbolic systems [22] and the Cattaneo law for heat diffusion [26, 6, 7]. In this approach, we only use vector variables instead of tensor variables. This allows to write the approximate systems in the standard form of balance laws. For this purpose, we decompose the diffusion term Δu into first-order derivative terms of u by introducing d full-rank matrices of order $d \times r$ with $r \geq d$. We prove the existence with explicit examples of these matrices in the cases where $r \geq d^2$ and $r = d$. In the latter case, Hurwitz-Radon matrices are concerned. See [33, 15, 1] for these matrices and relations with quadratic forms, and [18] for relations with the spin geometry and Clifford algebras.

More precisely, we consider the compressible Navier-Stokes equations under the form

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \nu \Delta u + \mu \nabla(\operatorname{div} u), \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \tag{1.8}$$

where the viscosity coefficients ν and μ are supposed to satisfy

$$\nu + \mu > 0 \ \text{ for } d = 1 \quad \text{and} \quad \nu > 0, \ \mu \geq 0 \ \text{ for } d \geq 2. \tag{1.9}$$

Now we show that system (1.1) with (1.2)-(1.3) is included in (1.8) with (1.9). Indeed, since

$$\operatorname{div}(\nabla u) = \Delta u, \quad \operatorname{div}(\nabla u)^T = \nabla(\operatorname{div} u),$$

we have

$$\operatorname{div}\left(\nu\sigma(u) + \lambda(\operatorname{div} u) I_d\right) = \begin{cases} \lambda \Delta u, & d = 1, \\ \nu \Delta u + \mu \nabla(\operatorname{div} u), & d \geq 2, \end{cases}$$

where

$$\mu = \nu + \lambda - \frac{2\nu}{d} \geq \lambda, \quad \text{for } d \geq 2.$$

Therefore, the conditions in (1.9) are satisfied if $\nu > 0$ and $\lambda > 0$. It is clear that system (1.8) in one space dimension is an easy case, because

$$\nu \Delta u + \mu \nabla(\operatorname{div} u) = (\nu + \mu) \partial_{xx} u, \quad x \in \mathbb{R}.$$

For simplifying the presentation of the problem, in what follows, we only consider (1.8) for $d \geq 2$.

Let $r \geq d$ be an integer and M_i $(1 \leq i \leq d)$ be real constant matrices of order $d \times r$ which satisfy

$$M_i M_i^T = I_d \ \text{ and } \ M_i M_j^T + M_j M_i^T = 0 \ \text{ for } 1 \leq i, j \leq d, \ j \neq i. \tag{1.10}$$

The existence of these matrices is given in Section 2 (see Proposition 2.1). Let $v^I \in \mathbb{R}^r$ and $v^{II} \in \mathbb{R}$ be auxiliary variables. We introduce the following first-order system of balance laws with relaxation :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \sqrt{\nu} \displaystyle\sum_{j=1}^{d} M_j \partial_{x_j} v^I + \sqrt{\mu}\, \nabla v^{II} = 0, \\ \varepsilon_1 \partial_t v^I + \sqrt{\nu} \displaystyle\sum_{j=1}^{d} M_j^T \partial_{x_j} u = -v^I, \\ \varepsilon_2 \partial_t v^{II} + \sqrt{\mu} \operatorname{div} u = -v^{II}, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \end{cases} \tag{1.11}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are relaxation times. System (1.11) is referred to as relaxed Euler system with vector variables. Formally, as $(\varepsilon_1, \varepsilon_2) \to 0$, the last two equations in (1.11) yield

$$v^I = -\sqrt{\nu} \sum_{j=1}^{d} M_j^T \partial_{x_j} u, \quad v^{II} = -\sqrt{\mu} \operatorname{div} u,$$

which implies that

$$\sqrt{\mu} \nabla v^{II} = -\mu \nabla(\operatorname{div} u).$$

Moreover, by (1.10), we have

$$\sqrt{\nu} \sum_{j=1}^{d} M_j \partial_{x_j} v^I = -\nu \Delta u.$$

Substituting these relations into the first two equations in (1.11), we get (1.8). This shows that the compressible Navier-Stokes equations (1.8) are the formal limit system of (1.11) as $(\varepsilon_1, \varepsilon_2) \to 0$. We point out that the first-order system (1.11) is not included in the class of systems studied in [31, 32].

The compressible Euler equations are part of (1.11) and such a structure allows us to observe easily the symmetrizable hyperbolicity of the system. In the absence of a theory on the global existence of weak solutions for the nonlinear hyperbolic system in several space dimensions, we consider smooth solutions in Sobolev spaces to the Cauchy problem for (1.11). For this purpose, we first construct a strictly convex entropy of the system. Remark that the existence of a strictly convex entropy is very important in our problem because it provides a symmetrizer of the system to obtain energy estimates in Sobolev spaces. It also provides an L^2 energy equality which is the first step in the study of the global existence of solutions.

We prove that system (1.11) converges to the Navier-Stokes equations not only in a uniform time interval but also globally for all time when the initial data are near constant equilibrium states. In this paper, the convergence is related to a limit as the relaxation times go to zero. By the local convergence, we mean that the smooth solution of an approximate system converges to a smooth solution of the limit system in a uniform time interval with a precise error estimate in $C([0,T]; H^m(\mathbb{R}^d))$, where $H^m(\mathbb{R}^d)$ is a usual Sobolev space. The global convergence means that, for all time, the sequence of smooth solutions of an approximate system admits a convergent subsequence (in strong or weak topology) whose limit is a global smooth solution of the limit system. The global convergence is based on the uniform global existence of solutions with respect to the relaxation times together with compactness arguments.

In the proof of the local convergence, we need to deal with initial layers for (v^I, v^{II}) by introducing correction terms. For simplifying the presentation, we take $\varepsilon_1 = \varepsilon_2$ as in [39]. The proof is based on energy estimates by choosing an appropriate symmetrizer of the system. In the result of the global convergence, we do not require any relation between ε_1 and ε_2, but the initial data are supposed to be in a uniform neighborhood of constant equilibrium states. The proof is based on three main steps. The first step concerns an L^2 estimate which follows from the entropy equality with a strictly convex entropy. The second step is to prove a usual higher order estimate with a dissipation estimate for (v^I, v^{II}) by using the symmetrizer mentioned above. The last step concerns a dissipation estimate for $(\nabla\rho, \nabla u)$, which depends strongly on condition (1.10) (see Lemmas 4.3-4.4 in Section 4).

The second goal of this paper is to justify the convergence of the relaxed Euler systems with tensor variables to the Navier-Stokes equations. The local convergence for (1.5) can be obtained in a similar way to that for the relaxed Euler systems with vector variables. The proof of the global convergence is given for both (1.5) and (1.7), for which we construct strictly convex entropies with entropy-flux. In the result of the global convergence for (1.7), ε_1 and ε_2 are independent. Besides the usual smallness conditions on the initial data, we also need a smallness condition on $\varepsilon_2 \mathrm{tr}(\pi_1)/\sqrt{\varepsilon_1}$ at the initial time and this condition disappears when $\varepsilon_2 = O(\varepsilon_1)$ or $\mathrm{tr}(\pi_1) = 0$ at $t = 0$ (see Theorem 6.2). We mention that in [39], a strictly convex entropy and the local convergence of (1.5) were established under a restriction condition $\mathrm{tr}(\pi_1) = 0$ at $t = 0$, which is removed in our result. This is a natural treatment since the expression of the entropy and entropy-flux should be independent of the initial data.

In the incompressible case with $d \geq 2$, the Navier-Stokes equations read

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \\ \mathrm{div}\, u = 0. \end{cases} \tag{1.12}$$

where $\cdot$ stands for the inner product in $\mathbb{R}^d$. We propose the relaxed Euler systems with vector variables

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p + \sqrt{\nu}\sum_{j=1}^{d} M_j \partial_{x_j} v = 0, \\ \varepsilon\partial_t v + \sqrt{\nu}\sum_{j=1}^{d} M_j^T \partial_{x_j} u = -v, \\ \operatorname{div} u = 0, \end{cases} \tag{1.13}$$

and that with tensor variables

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p + \operatorname{div}\pi = 0, \\ \varepsilon\partial_t \pi + \nu\big(\nabla u + (\nabla u)^T\big) = -\pi, \\ \operatorname{div} u = 0. \end{cases} \tag{1.14}$$

We prove that both systems (1.13) and (1.14) converge to (1.12) in the same framework as above.

Finally, we remark that the idea of this kind of approximations of a second-order partial differential equation by first-order hyperbolic systems is not recent. It comes back to the study by Maxwell and Cattaneo [26, 6, 7]. Here is a simple example. The heat equation $\partial_t u - \Delta u = 0$ can be expressed by the first law of thermodynamics $\partial_t u + \operatorname{div} q = 0$ together with Fourier law $q = -\nabla u$. Cattaneo proposed a revised law $\varepsilon\partial_t q + \nabla u = -q$, called now Cattaneo law or Maxwell-Cattaneo law, where $\varepsilon > 0$ is a relaxation time. This forms a linear hyperbolic system with relaxation

$$\begin{cases} \partial_t u + \operatorname{div} q = 0, \\ \varepsilon\partial_t q + \nabla u = -q, \end{cases}$$

and we recover the heat equation as $\varepsilon \to 0$. This idea was developed later in the approximation of nonlinear second-order systems of partial differential equations by first-order hyperbolic systems. We refer, for instance, to [36, 28, 5] for the approximation of the incompressible Navier-Stokes equations by using Oldroyd-type constitutive laws with relaxation, to [10, 35] for the approximation of the Timoshenko-Fourier system by the Timoshenko-Cattaneo system, and to [13, 14] for the local convergence of hyperbolic-parabolic systems to the full compressible Navier-Stokes equations. See also [3, 30, 27, 34] on this topic.

This paper is organized as follows. In the next section, we construct the relaxed Euler systems with vector variables based on decompositions of Δu. We study the symmetrizable hyperbolicity of the systems and recall results on the local existence of smooth solutions. In Section 3 we prove the local convergence of the systems. Section 4 is devoted to the result and the proof of the global convergence of the systems. In Section 5, we consider the relaxed Euler systems in the incompressible case and prove their convergence to the incompressible Navier-Stokes equations. In the last section, we study the relaxed Euler systems with tensor variables and prove similar results to those for the relaxed Euler systems with vector variables.

2. Relaxed Euler systems

2.1. **Decomposition of Δu.**

The construction of relaxed Euler systems is based on the decomposition of Δ into two first-order differential operators. Let $u : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a smooth function. In the usual decomposition $\Delta u = \operatorname{div}(\nabla u)$, ∇u is a tensor for $d \geq 2$. In order to use the vector variables instead of the tensor variables, we employ different decompositions of Δu as follows.

Let $r \geq d$ be an integer and M_i $(1 \leq i \leq d)$ be real constant matrices of order $d \times r$ which satisfy

$$M_i M_i^T = I_d \ \text{ and } \ M_i M_j^T + M_j M_i^T = 0 \ \text{ for } 1 \leq i, j \leq d, \ j \neq i. \tag{2.1}$$

The decomposition of Δu is

$$\Delta u = \sum_{i,j=1}^{d} M_i M_j^T \partial^2_{x_i x_j} u = \sum_{i=1}^{d} M_i \partial_{x_i} \left(\sum_{j=1}^{d} M_j^T \partial_{x_j} u \right). \tag{2.2}$$

The first condition in (2.1) implies that M_i is a full-rank matrix for all $1 \leq i \leq d$. If the entries of M_i are regarded as unknown variables, (2.1) represents at most $(d+1)d^3/2$ independent equations with $d^2 r$ unknown variables. We observe that for a fixed pair (d, r), if (2.1) admits a solution M_i $(1 \leq i \leq d)$, then for all $\tilde{r} > r$, the matrices $\tilde{M}_i$ $(1 \leq i \leq d)$ (of order $d \times \tilde{r}$) defined by $\tilde{M}_i = (M_i, 0_{d\times(\tilde{r}-r)})$ still satisfy (2.1), where $0_{d\times s}$ is the zero matrix of order $d \times s$ for some integer $s \in \mathbb{N}$. The solution for (2.1) with $d = 1$ is obvious (take $r = 1$). For $d \geq 2$, we consider the following two cases where Case 1 shows that the solution of (2.1) exists for all $r \geq d^2$ and Case 2 shows that the solution with square matrices of (2.1) exists only when $d = 1, 2, 4, 8$.

Case 1 : $r \geq d^2$. These matrices can be constructed explicitly as follows.

Let O_i $(1 \leq i \leq d)$ be any orthogonal matrices of order d. We take

$$M_i = \big(0_{d\times(i-1)d}, O_i, 0_{d\times(r-id)}\big), \quad 1 \leq i \leq d.$$

Since $M_i M_j^T = \delta_{ij} I_d$, these matrices fulfill all conditions in (2.1). In particular, when $r = d^2$ and $O_i = I_d$ for all $1 \leq i \leq d$, easy calculations give

$$\sum_{j=1}^{d} M_j^T \partial_{x_j} u = \begin{pmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_d} u \end{pmatrix} \overset{def}{=} \nabla_b u.$$

Moreover, let

$$v^I = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \quad \text{with } \ v_i(t,x) \in \mathbb{R}^d, \quad 1 \leq i \leq d.$$

Then

$$M_i v^I = v_i \quad \text{and} \quad \sum_{i=1}^{d} M_i \partial_{x_i} v^I = \sum_{i=1}^{d} \partial_{x_i} v_i.$$

The latter is the divergence of v^I by blocks. In this case, the third equation in (1.11) becomes

$$\varepsilon_1 \partial_t v^I + \sqrt{\nu} \nabla_b u = -v^I,$$

or equivalently,

$$\varepsilon_1 \partial_t v_i + \sqrt{\nu} \partial_{x_i} u = -v_i, \quad 1 \leq i \leq d.$$

Case 2 : $r = d$. This is an interesting case where each M_i is a square matrix and also of the minimum size. The conditions in (2.1) mean that M_i is orthogonal and $M_i M_j^T$ is anti-symmetric for all $1 \leq i, j \leq d$ and $j \neq i$. When $d \geq 3$ is odd, it is clear that there don't exist such square matrices. Indeed, the second condition in (2.1) implies that

$$\det(M_i)\det(M_j) = 0, \quad j \neq i,$$

which is contradictory to the fact that M_i is an orthogonal matrix.

When d is even, it is possible to build M_i as follows. Let us fix an orthogonal matrix M_d, denoted by O. Define $A_i = M_i O^T$ for $1 \leq i \leq d-1$. When A_i is given, we obtain M_i through $M_i = A_i O$ for all $1 \leq i \leq d-1$ and $M_d = O$.

It follows that A_i is anti-symmetric and (2.1) is equivalent to

$$A_iA_i^T = I_d, \quad A_i^2 = -I_d, \quad A_iA_j^T + A_jA_i^T = 0 \tag{2.3}$$

for all $1 \leq i, j \leq d-1,\ j \neq i$. Let $s \geq 1$ be an integer. Recall that square matrices A_1, $A_2, \cdots, A_s$ of order d are Hurwitz-Radon matrices if (2.3) is satisfied for all $1 \leq i, j \leq s,\ j \neq i$. Following the results in [33, 15], for d given, the Hurwitz-Radon matrices can be constructed up to a Radon number $s = \rho(d)$. Moreover, these matrices can be written with entries 0 and ± 1 alone [8]. More precisely, all integer $d \geq 1$ can be expressed as

$$d = (2c+1)2^{4a+b}, \quad a, b, c \in \mathbb{N}, \quad 0 \leq b \leq 3.$$

Then the Radon number $\rho(d)$ is given by [33]

$$\rho(d) = 8a + 2^b - 1.$$

When $4a + b = 0$, we have $a = b = 0$. Then d is odd and $\rho(d) = 0$. Thus, $4a + b \geq 1$ is a necessary condition to obtain a minimum positive number of $\rho(d)$. An explicit construction of Hurwitz-Radon matrices can be found in [8].

Here we are interested in determining d such that $\rho(d) \geq d - 1$. This allows to obtain A_1, $A_2, \cdots, A_{d-1}$ which provide M_1, $M_2, \cdots, M_d$. By the comparison of the expressions of d and $\rho(d)$ above, we have

$$d - 1 - \rho(d) = (c2^{4a+b} + 2^{4a-1+b} - 8a) + (c2^{4a} + 2^{4a-1} - 1)2^b.$$

Hence, it is easy to see that $d - 1 > \rho(d)$ when $a \geq 1$ or $c \geq 1$. Nevertheless, in the other case, namely, $a = c = 0$, we have exactly $\rho(d) = d - 1 = 2^b - 1$. Thus, A_1, $A_2, \cdots, A_{d-1}$ can be constructed for all dimension d of the form $d = 2^b$ with $b = 0, 1, 2, 3$, namely, $d = 1, 2, 4, 8$.

We conclude the results from the above discussion in the following proposition.

Proposition 2.1. *Let $d \geq 1$ be a given integer. There exist integer $r \geq d$ and matrices $M_i\,(1 \leq i \leq d)$ of order $d \times r$ satisfying (2.1) in the following two cases :*
i) for all $r \geq d^2$,
ii) for $r = d$ with $d = 1, 2, 4, 8$.

Now we give examples of matrices $M_i\,(1 \leq i \leq d)$ satisfying (2.1) for $r = d$ with $d = 2, d = 4$. Let Q be the matrix defined by

$$Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easy to see that Q is orthogonal and anti-symmetric. For $d = 2$, it suffices to take

$$M_1 = Q, \quad M_2 = I_2.$$

For $d = 4$, we define these matrices by blocs based on Q and I_2,

$$M_1 = \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}, \quad M_4 = I_4,$$

where M_1, M_2 and M_3 are orthogonal and anti-symmetric.

2.2. **Relaxed Euler systems and hyperbolicity.**

As mentioned in the introduction, the relaxed Euler systems with vector variables are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \sqrt{\nu} \sum_{j=1}^{d} M_j \partial_{x_j} v^I + \sqrt{\mu}\, \nabla v^{II} = 0, \\ \varepsilon_1 \partial_t v^I + \sqrt{\nu} \sum_{j=1}^{d} M_j^T \partial_{x_j} u = -v^I, \\ \varepsilon_2 \partial_t v^{II} + \sqrt{\mu}\, \operatorname{div} u = -v^{II}, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d. \end{cases} \tag{2.4}$$

Now we study the hyperbolicity of the systems. Let $D_0(\varepsilon)$ be a diagonal matrix of order $r+1$ and N_j be a matrix of order $d \times (r+1)$ defined by

$$D_0(\varepsilon) = \operatorname{diag}(\varepsilon_1 I_r, \varepsilon_2), \quad N_j = \big(\sqrt{\nu} M_j, \sqrt{\mu} e_j\big),$$

where $(e_1, \cdots, e_d)$ is the standard basis of $\mathbb{R}^d$. Since

$$\operatorname{div} u = \sum_{j=1}^{d} e_j^T \partial_{x_j} u, \quad \nabla v^{II} = \sum_{j=1}^{d} e_j \partial_{x_j} v^{II},$$

system (2.4) is equivalent to

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \rho \partial_t u + (\rho u \cdot \nabla) u + \nabla p(\rho) + \sum_{j=1}^{d} N_j \partial_{x_j} v = 0, \\ D_0(\varepsilon) \partial_t v + \sum_{j=1}^{d} N_j^T \partial_{x_j} u = -v, \quad v = \begin{pmatrix} v^I \\ v^{II} \end{pmatrix}. \end{cases} \tag{2.5}$$

A part of (2.5) is the compressible Euler equations which is a symmetrizable hyperbolic system. We can choose a diagonal matrix to be its symmetrizer [22]. Since $D_0(\varepsilon)$ is also a diagonal matrix, by the position of N_j and N_j^T, we see easily that (2.5) is a symmetrizable hyperbolic system too.

For $\rho > 0$, let h be the enthalpy function defined by $h'(\rho) = p'(\rho)/\rho$. System (2.4) can be further written as

$$D_1(\varepsilon) \partial_t U + \sum_{j=1}^{d} A_j(\rho, u) \partial_{x_j} U = S(v), \quad U = \begin{pmatrix} \rho \\ u \\ v \end{pmatrix} \in \mathbb{R}^{2+d+r}, \tag{2.6}$$

where

$$D_1(\varepsilon) = \operatorname{diag}(I_{d+1}, \varepsilon_1 I_r, \varepsilon_2), \quad S(v) = -\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix},$$

$$A_j(\rho, u) = \begin{pmatrix} u_j & \rho e_j^T & 0 \\ h'(\rho) e_j & u_j I_d & \frac{1}{\rho} N_j \\ 0 & N_j^T & 0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}.$$

Let (E_0, F_0) be the pair of entropy-entropy flux for the Euler equations, given by

$$\begin{cases} E_0(\rho,u) = \dfrac{1}{2}\rho|u|^2 + H(\rho), \\ F_0(\rho,u) = \dfrac{1}{2}\rho|u|^2 u + \rho h(\rho)u, \end{cases} \tag{2.7}$$

where $H'(\rho) = h(\rho)$ and $|\cdot|$ is the usual Euclidean norm. It is known that E_0 is strictly convex with respect to the conservative variable $(\rho, \rho u)$ of the Euler equations for $\rho > 0$. We define functions E and F by

$$\begin{cases} E(U) = E_0(\rho,u) + \dfrac{\varepsilon_1}{2}|v^I|^2 + \dfrac{\varepsilon_2}{2}|v^{II}|^2, \\ F(U) = F_0(\rho,u) + a_N(u,v), \end{cases} \tag{2.8}$$

where $a_N(u,v) : \mathbb{R}^d \times \mathbb{R}^{r+1} \longrightarrow \mathbb{R}^d$ is a bilinear application defined by

$$a_N(u,v) = \begin{pmatrix} u^T N_1 v \\ \vdots \\ u^T N_d v \end{pmatrix}.$$

We check easily that a smooth solution U of (2.5) satisfies the energy equality

$$\partial_t E(U) + \operatorname{div} F(U) + |v|^2 = 0. \tag{2.9}$$

Therefore, (E, F) is a pair of entropy-entropy flux. Since E is a strictly convex function with respect to the conservative variable $(\rho, \rho u, v)$ for $\rho > 0$, E is a strictly convex entropy of the system. By results in [12, 11, 2], this implies again that system (2.5) is symmetrizable hyperbolic.

2.3. **Local existence of solutions.**

Let us denote $\varepsilon = (\varepsilon_1, \varepsilon_2)$. We consider the Cauchy problem for the relaxed Euler system :

$$\begin{cases} D_1(\varepsilon)\partial_t U^\varepsilon + \displaystyle\sum_{j=1}^{d} A_j(\rho^\varepsilon, u^\varepsilon)\partial_{x_j} U^\varepsilon = S(v^\varepsilon), \\ t = 0 \ : \ (\rho^\varepsilon, u^\varepsilon, v^\varepsilon) = (\rho_0^\varepsilon, u_0^\varepsilon, v_0^\varepsilon), \end{cases} \quad U^\varepsilon = \begin{pmatrix} \rho^\varepsilon \\ u^\varepsilon \\ v^\varepsilon \end{pmatrix}, \tag{2.10}$$

where the initial data may depend on ε. For $m \in \mathbb{N}$, we denote by H^m the Sobolev space $H^m(\mathbb{R}^d)$ and by $\|\cdot\|_m$ its usual norm. Let $m > \dfrac{d}{2} + 1$ be an integer. The equilibrium state we take is $(1, 0, 0)$ for (ρ, u, v). We assume $(\rho_0^\varepsilon - 1, u_0^\varepsilon, v_0^{I,\varepsilon}, v_0^{II,\varepsilon}) \in H^m$ with $\inf\limits_{x\in\mathbb{R}^d} \rho_0^\varepsilon(x) > 0$. By the local existence of smooth solutions for symmetrizable hyperbolic systems (see [19, 17, 22]), there exist a maximal time $T_0^\varepsilon > 0$ possibly depending on ε and a unique smooth solution U^ε to (2.10), defined on time interval $[0, T_0^\varepsilon)$. This solution satisfies

$$(\rho^\varepsilon - 1, u^\varepsilon, v^{I,\varepsilon}, v^{II,\varepsilon}) \in C([0,T_0^\varepsilon); H^m) \cap C^1\big([0,T_0^\varepsilon); H^{m-1}\big).$$

In the case of the Cauchy problem for the compressible Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \nu\Delta u + \mu\nabla(\operatorname{div} u), \\ t = 0 \ : \ (\rho, u) = \big(\rho_0, u_0\big), \end{cases} \tag{2.11}$$

the local existence of solutions is analogous (see [16, 25]). Assume $(\rho_0 - 1, u_0) \in H^m$ with $\inf\limits_{x\in\mathbb{R}^d} \rho_0(x) > 0$. There exist a time $T_0 > 0$ and a unique smooth solution (ρ, u) to (2.11), defined

on time interval $[0,T_0]$. This solution satisfies

$$(\rho-1,u)\in C([0,T_0];H^m)\cap C^1\big([0,T_0];H^{m-1}\big),\quad \nabla u\in L^2(0,T_0;H^m),\quad \inf_{(t,x)\in[0,T_0]\times\mathbb{R}^d}\rho(t,x)>0.$$

For the local convergence, where $\varepsilon_1=\varepsilon_2$, still denoted by ε, we have $D_0(\varepsilon)=\varepsilon I_d$. We define

$$\bar v=-\sum_{j=1}^d N_j^T\partial_{x_j}u,\quad \bar v_0=-\sum_{j=1}^d N_j^T\partial_{x_j}u_0,\quad v_0^\varepsilon=\begin{pmatrix} v_0^{I,\varepsilon}\\ v_0^{II,\varepsilon}\end{pmatrix}. \tag{2.12}$$

Let v_0 be the smooth function satisfying $\lim_{\varepsilon\to0}v_0^\varepsilon=v_0$ in a strong topology. If $\bar v_0\neq v_0$, because of initial layer formations near $t=0$, it is impossible that $(v^\varepsilon)_{\varepsilon>0}$ converges to $\bar v$ uniformly in a time interval $[0,T]$ with $T>0$. To treat this difficulty, we introduce a correction variable v_L depending on (s,x) with $s=t/\varepsilon$. In view of the equation for v^ε in (2.5), we define v_L by

$$\partial_s v_L=-v_L,$$

which gives

$$v_L(s,x)=v_L(0,x)e^{-s}.$$

We hope that $(v^\varepsilon-v_L)_{\varepsilon>0}$ converges to $\bar v$ uniformly in a time interval $[0,T]$. In particular, this implies that $(v^\varepsilon-v_L)(0,\cdot)_{\varepsilon>0}$ converges to $\bar v(0,\cdot)$, hence

$$v_L(0,x)=v_0(x)-\bar v_0(x).$$

From now on, we denote

$$v_\varepsilon(t,x)=\big(v_0(x)-\bar v_0(x)\big)e^{-t/\varepsilon}, \tag{2.13}$$

which satisfies

$$\varepsilon\partial_t v_\varepsilon=-v_\varepsilon.$$

It follows from (2.12) that

$$\nu\Delta u+\mu\nabla(\operatorname{div}u)=-\sum_{j=1}^d N_j\partial_{x_j}(\bar v+v_\varepsilon)+\sum_{j=1}^d N_j\partial_{x_j}v_\varepsilon$$

and

$$\varepsilon\partial_t(\bar v+v_\varepsilon)+\sum_{j=1}^d N_j^T\partial_{x_j}u=-(\bar v+v_\varepsilon)+\varepsilon\partial_t\bar v.$$

Combining these two equations with (2.11) yields

$$\begin{cases} D_1(\varepsilon)\partial_t U_\varepsilon+\displaystyle\sum_{j=1}^d A_j(\rho,u)\partial_{x_j}U_\varepsilon=S(\bar v+v_\varepsilon-\varepsilon\partial_t\bar v)-\frac{1}{\rho}R(\nabla v_\varepsilon),\\ t=0\ :\ (\rho,u,\bar v+v_\varepsilon)=\big(\rho_0,u_0,v_0\big),\end{cases}\quad U_\varepsilon=\begin{pmatrix}\rho\\ u\\ \bar v+v_\varepsilon\end{pmatrix}, \tag{2.14}$$

where

$$R(\nabla v_\varepsilon)=-\begin{pmatrix}0\\ \displaystyle\sum_{j=1}^d N_j\partial_{x_j}v_\varepsilon\\ 0\end{pmatrix}.$$

3. Local convergence

In the results stated below, we suppose that $\nu > 0$, $\mu \geq 0$, and $M_i\,(1 \leq i \leq d)$ are any real constant matrices of order $d \times r$ satisfying (2.1). From this section, we denote by $c > 0$ and $c_i > 0\,(i \in \mathbb{N})$ generic constants independent of any time and ε. In the proof of theorems, we also denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and the usual norm in $L^2(\mathbb{R}^d)$, respectively. We often use the continuous embedding from $H^m(\mathbb{R}^d)$ to $W^{1,\infty}(\mathbb{R}^d)$ for $m > \dfrac{d}{2} + 1$.

Theorem 3.1. *(Local convergence for the relaxed Euler system)*

Let $\varepsilon_1 = \varepsilon_2 \overset{def}{=} \varepsilon > 0$. Let $m > \dfrac{d}{2} + 1$ be an integer. Let $(\rho_0^\varepsilon - 1, u_0^\varepsilon, v_0^{I,\varepsilon}, v_0^{II,\varepsilon}) \in H^m$, $(\rho_0 - 1, u_0) \in H^{m+2}$ and $v_0 \in H^{m+1}$. We assume $\inf\limits_{x \in \mathbb{R}^d} \rho_0(x) > 0$ and

$$\|\rho_0^\varepsilon - \rho_0\|_m + \|u_0^\varepsilon - u_0\|_m + \sqrt{\varepsilon}\,\|v_0^\varepsilon - v_0\|_m \leq c_1 \varepsilon, \tag{3.1}$$

where $c_1 > 0$ is a constant independent of ε.

Let (ρ, u) be the unique solution to (2.11) on $[0, T_0]$. Then there exists a constant $\varepsilon_0 \in (0, 1]$ depending on T_0, such that for all $\varepsilon \in (0, \varepsilon_0]$, the unique solution $(\rho^\varepsilon, u^\varepsilon, v^\varepsilon)$ to (2.10) is defined on $[0, T_0]$. Moreover,

$$\|\rho^\varepsilon(t) - \rho(t)\|_m^2 + \|u^\varepsilon(t) - u(t)\|_m^2 + \int_0^t \|v^\varepsilon(t') - \bar{v}(t') - v_\varepsilon(t')\|_m^2 dt' \leq c\varepsilon^2, \ \ \forall\, t \in [0, T_0], \tag{3.2}$$

$$\|v^\varepsilon(t) - \bar{v}(t) - v_\varepsilon(t)\|_m \leq c\sqrt{\varepsilon}, \quad \forall\, t \in [0, T_0], \tag{3.3}$$

where $\bar{v}$ and v_ε are defined in (2.12)-(2.13).

Proof. Let us introduce

$$z^\varepsilon = v^\varepsilon - (\bar{v} + v_\varepsilon), \ \ W_1^\varepsilon = \begin{pmatrix} \rho^\varepsilon - \rho \\ u^\varepsilon - u \end{pmatrix}, \ \ W^\varepsilon = \begin{pmatrix} W_1^\varepsilon \\ z^\varepsilon \end{pmatrix}, \ \ W_0^\varepsilon = \begin{pmatrix} \rho_0^\varepsilon - \rho_0 \\ u_0^\varepsilon - u_0 \\ v_0^\varepsilon - v_0 \end{pmatrix},$$

where v_ε is defined in (2.13). Obviously,

$$z^\varepsilon(0, \cdot) = v_0^\varepsilon - v_0.$$

Let $T^\varepsilon = \min(T_0^\varepsilon, T_0) \in (0, T_0]$. Then both systems in (2.10) and (2.14) are well defined in $[0, T^\varepsilon) \times \mathbb{R}^d$. Noting that $W^\varepsilon = U^\varepsilon - U_\varepsilon$, subtracting (2.10) and (2.14) yields

$$\begin{cases} D_1(\varepsilon)\partial_t W^\varepsilon + \sum\limits_{j=1}^d A_j(\rho^\varepsilon, u^\varepsilon)\partial_{x_j} W^\varepsilon = \ \sum\limits_{j=1}^d \big(A_j(\rho, u) - A_j(\rho^\varepsilon, u^\varepsilon)\big)\partial_{x_j} U_\varepsilon \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + S(z^\varepsilon + \varepsilon\partial_t \bar{v}) + \dfrac{1}{\rho} R(\nabla v_\varepsilon), \\ \qquad\qquad\qquad t = 0 \ : \ W^\varepsilon = W_0^\varepsilon, \end{cases} \tag{3.4}$$

for $(t, x) \in [0, T^\varepsilon) \times \mathbb{R}^d$. Since S is linear, we have

$$S(z^\varepsilon + \varepsilon\partial_t \bar{v}) = S(z^\varepsilon) + \varepsilon S(\partial_t \bar{v}).$$

Let $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq m$. we denote

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \quad \text{with} \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

We also denote

$$A_0(\rho) = \operatorname{diag}\left(h'(\rho), \rho I_d, I_{r+1}\right), \quad \tilde{A}_j(\rho, u) = A_0(\rho) A_j(\rho, u),$$

Then

$$A_0(\rho)D_1(\varepsilon)=\mathrm{diag}\left(h'(\rho),\rho I_d,\varepsilon I_{r+1}\right),\quad \tilde{A}_j(\rho,u)=\begin{pmatrix} h'(\rho)u_j & p'(\rho)e_j^T & 0\\ p'(\rho)e_j & \rho u_j I_d & N_j\\ 0 & N_j^T & 0\end{pmatrix}.$$

Since $A_0(\rho)D_1(\varepsilon)$ is symmetric positive definite and $\tilde{A}_j(\rho,u)$ is symmetric for all $1\le j\le d$, $A_0(\rho)D_1(\varepsilon)$ is a symmetrizer of system (2.6).

Applying ∂_x^α to (3.4), we get

$$\begin{aligned} &D_1(\varepsilon)\partial_t(\partial_x^\alpha W^\varepsilon)+\sum_{j=1}^d A_j(\rho^\varepsilon,u^\varepsilon)\partial_{x_j}(\partial_x^\alpha W^\varepsilon)\\ =\;& S(\partial_x^\alpha z^\varepsilon)+\varepsilon S(\partial_x^\alpha\partial_t\bar{v})+\partial_x^\alpha[\rho^{-1}R(\nabla v_\varepsilon)]+\sum_{j=1}^d I_j^\alpha, \end{aligned}\tag{3.5}$$

where

$$I_j^\alpha=A_j(\rho^\varepsilon,u^\varepsilon)\partial_x^\alpha(\partial_{x_j}W^\varepsilon)-\partial_x^\alpha\big(A_j(\rho^\varepsilon,u^\varepsilon)\partial_{x_j}W^\varepsilon\big)+\partial_x^\alpha\big[\big(A_j(\rho,u)-A_j(\rho^\varepsilon,u^\varepsilon)\big)\partial_{x_j}U_\varepsilon\big].$$

Taking the inner product of (3.5) with $2A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon$ in $L^2(\mathbb{R}^d)$ and using the fact that both matrices $A_0(\rho^\varepsilon)D_1(\varepsilon)$ and $\tilde{A}_j(\rho^\varepsilon,u^\varepsilon)$ are symmetric, we obtain the classical energy equality [22] :

$$\begin{aligned} \frac{d}{dt}\big\langle A_0(\rho^\varepsilon)D_1(\varepsilon)\partial_x^\alpha W^\varepsilon,\partial_x^\alpha W^\varepsilon\big\rangle &= 2\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,S(\partial_x^\alpha z^\varepsilon)\big\rangle+2\varepsilon\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,S(\partial_x^\alpha\partial_t\bar{v})\big\rangle\\ &\quad+2\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,\partial_x^\alpha[\rho^{-1}R(\nabla v_\varepsilon)]\big\rangle\\ &\quad+2\sum_{j=1}^d\big\langle A_0(\rho^\varepsilon)I_j^\alpha,\partial_x^\alpha W^\varepsilon\big\rangle+\big\langle\,\mathrm{div}\,\vec{A}(\rho^\varepsilon,u^\varepsilon)\partial_x^\alpha W^\varepsilon,\partial_x^\alpha W^\varepsilon\big\rangle, \end{aligned}\tag{3.6}$$

where

$$\mathrm{div}\,\vec{A}(\rho,u)=\partial_t A_0(\rho)D_1(\varepsilon)+\sum_{j=1}^d\partial_{x_j}\tilde{A}_j(\rho,u).\tag{3.7}$$

Since $\inf\limits_{x\in\mathbb{R}^d}\rho_0(x)>0$, by (3.1), we may first suppose that, for sufficiently small ε,

$$\|W^\varepsilon(t)\|_m\le c,\quad \rho^\varepsilon(t)\ge c_0>0,\quad \forall\, t\in[0,T^\varepsilon).\tag{3.8}$$

In view of the expression of A_0, $D_1(\varepsilon)$ and S, it is straightforward that

$$\begin{aligned} c_1\big(\|\partial_x^\alpha W_1^\varepsilon\|^2+\varepsilon\|\partial_x^\alpha z^\varepsilon\|^2\big) &\le \big\langle A_0(\rho^\varepsilon)D_1(\varepsilon)\partial_x^\alpha W^\varepsilon,\partial_x^\alpha W^\varepsilon\big\rangle\\ &\le c\big(\|\partial_x^\alpha W_1^\varepsilon\|^2+\varepsilon\|\partial_x^\alpha z^\varepsilon\|^2\big), \end{aligned}\tag{3.9}$$

$$2\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,S(\partial_x^\alpha z^\varepsilon)\big\rangle=-2\|\partial_x^\alpha z^\varepsilon\|^2,\tag{3.10}$$

and

$$2\varepsilon\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,S(\partial_x^\alpha\partial_t\bar{v})\big\rangle=-2\varepsilon\big\langle\partial_x^\alpha z^\varepsilon,\partial_x^\alpha\partial_t\bar{v}\big\rangle.$$

By the Young inequality, we have

$$2\big|\varepsilon\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon,S(\partial_x^\alpha\partial_t\bar{v})\big\rangle\big|\le\|\partial_x^\alpha z^\varepsilon\|^2+c\varepsilon^2\|\partial_t\bar{v}\|_m^2.\tag{3.11}$$

Next, by the Cauchy-Schwarz inequality together with (2.13) , we have

$$\begin{aligned} 2\big\langle A_0(\rho^\varepsilon)\partial_x^\alpha W^\varepsilon, \partial_x^\alpha[\rho^{-1}R(\nabla v_\varepsilon)]\big\rangle &= -2\sum_{j=1}^d \big\langle \rho^\varepsilon \partial_x^\alpha(u^\varepsilon - u), N_j\partial_x^\alpha(\rho^{-1}\partial_{x_j}v_\varepsilon)\big\rangle \\ &\le c\|W_1^\varepsilon\|_m\|v_0-\bar v_0\|_{m+1}e^{-\frac{t}{\varepsilon}}. \end{aligned} \tag{3.12}$$

For the term containing I_j^α, we observe that the last $r+1$ lines of $A_j(\rho,u)$ are constant. This implies that the last $r+1$ lines of $A_j(\rho,u)-A_j(\rho^\varepsilon,u^\varepsilon)$ vanish. By the Moser-type calculus inequalities (see Proposition 2.1 in [22], p.43), we have

$$\big|\langle A_0(\rho^\varepsilon)\partial_x^\alpha\big[\big(A_j(\rho,u)-A_j(\rho^\varepsilon,u^\varepsilon)\big)\partial_{x_j}U_\varepsilon\big], \partial_x^\alpha W^\varepsilon\rangle\big| \le c\|W_1^\varepsilon\|_m^2.$$

Similarly,

$$\big|\langle A_0(\rho^\varepsilon)\big[A_j(\rho,u)\partial_x^\alpha(\partial_{x_j}W^\varepsilon) - \partial_x^\alpha\big(A_j(\rho^\varepsilon,u^\varepsilon)\partial_{x_j}W^\varepsilon\big)\big], \partial_x^\alpha W^\varepsilon\rangle\big| \le c\|W_1^\varepsilon\|_m^2.$$

Therefore,

$$2\sum_{j=1}^d \big|\big\langle A_0(\rho^\varepsilon)I_j^\alpha, \partial_x^\alpha W^\varepsilon\big\rangle\big| \le c\|W_1^\varepsilon\|_m^2. \tag{3.13}$$

For the last term in (3.6), from the definition of A_0, D_1 and $\tilde A_j$, $\operatorname{div}\vec A(\rho^\varepsilon,u^\varepsilon)$ can be expressed as

$$\operatorname{div}\vec A(\rho^\varepsilon,u^\varepsilon) = \operatorname{diag}(A_{11}^\varepsilon,0), \quad A_{11}^\varepsilon = O\big(\nabla(\rho^\varepsilon,u^\varepsilon)\big),$$

where A_{11}^ε is a square matrix of order $d+1$. It follows that

$$\big|\big\langle \operatorname{div}\vec A(\rho^\varepsilon,u^\varepsilon)\partial_x^\alpha W^\varepsilon, \partial_x^\alpha W^\varepsilon\big\rangle\big| \le c\|\partial_x^\alpha W_1^\varepsilon\|^2. \tag{3.14}$$

Thus, we conclude from (3.6) and (3.10)-(3.14) that

$$\begin{aligned} &\frac{d}{dt}\big\langle A_0(\rho^\varepsilon)D_1(\varepsilon)\partial_x^\alpha W^\varepsilon, \partial_x^\alpha W^\varepsilon\big\rangle + 2\|\partial^\alpha z^\varepsilon\|^2 \\ \le\ & c\varepsilon^2\|\partial_t\bar v\|_m^2 + c\|W_1^\varepsilon\|_m^2 + c\|W_1^\varepsilon\|_m\|v_0-\bar v_0\|_{m+1}e^{-\frac{t}{\varepsilon}}. \end{aligned}$$

Integrating this inequality over $[0,t]$ with $t\in[0,T^\varepsilon)$ and adding the inequalities for all α with $|\alpha|\le m$, together with (3.9), it yields

$$\begin{aligned} &\|W_1^\varepsilon(t)\|_m^2 + \varepsilon\|z^\varepsilon(t)\|_m^2 + \int_0^t \|z^\varepsilon(t')\|_m^2dt' \\ \le\ & \|W_1^\varepsilon(0)\|_m^2 + \varepsilon\|z^\varepsilon(0)\|_m^2 + c\varepsilon^2\int_0^t\|\partial_t\bar v(t')\|_m^2dt' \\ &+ c\int_0^t\|W_1^\varepsilon(t')\|_m^2dt' + c\int_0^t\|W_1^\varepsilon(t')\|_m\|v_0-\bar v_0\|_{m+1}e^{-\frac{t'}{\varepsilon}}dt'. \end{aligned}$$

From (3.1), we have

$$\|W_1^\varepsilon(0)\|_m^2 + \varepsilon\|z^\varepsilon(0)\|_m^2 \le c\varepsilon^2.$$

On the other hand, using (2.12) and

$$-\partial_t u = (u\cdot\nabla)u + \nabla h(\rho) - \frac{1}{\rho}\big(\nu\Delta u + \mu\nabla(\operatorname{div}u)\big),$$

we also have

$$\partial_t\bar v = \sum_{j=1}^d N_j^T\partial_{x_j}\Big((u\cdot\nabla)u + \nabla h(\rho) - \frac{1}{\rho}\big(\nu\Delta u + \mu\nabla(\operatorname{div}u)\big)\Big).$$

Hence,

$$\|\partial_t \bar{v}\|_m \le c\big(\|\nabla\rho\|_{m+1} + \|\nabla u\|_{m+2}\big).$$

Since $(\rho_0 - 1, u_0) \in H^{m+2}$ and $v_0 \in H^{m+1}$, we have $v_0 - \bar{v}_0 \in H^{m+1}$ and the solution to (2.11) satisfies $\nabla\rho \in C([0,T_0]; H^{m+1})$ and $\nabla u \in L^2(0,T_0; H^{m+2})$. It follows that

$$\|W_1^\varepsilon(t)\|_m^2 + \varepsilon\|z^\varepsilon(t)\|_m^2 + \int_0^t \|z^\varepsilon(t')\|_m^2 dt' \le c\int_0^t \|W_1^\varepsilon(t')\|_m^2 dt' + c\int_0^t \|W_1^\varepsilon(t')\|_m e^{-\frac{t'}{\varepsilon}} dt' + c\varepsilon^2.$$

Let

$$y(t) = \left(c\int_0^t \|W_1^\varepsilon(t')\|_m^2 dt' + c\int_0^t \|W_1^\varepsilon(t')\|_m e^{-\frac{t'}{\varepsilon}} dt' + c\varepsilon^2 \right)^{\frac{1}{2}}.$$

Then

$$\|W_1^\varepsilon(t)\|_m \le y(t),$$

and

$$y'(t) \le cy(t) + ce^{-\frac{t}{\varepsilon}}, \quad y(0) = c\varepsilon.$$

Noting that $T^\varepsilon \le T_0$ and

$$\int_0^t e^{-\frac{t'}{\varepsilon}} dt' \le \varepsilon,$$

by a Gronwall inequality, we obtain $y(t) \le c\varepsilon$, which implies that

$$\|W_1^\varepsilon(t)\|_m^2 + \varepsilon\|z^\varepsilon(t)\|_m^2 + \int_0^t \|z^\varepsilon(t')\|_m^2 dt' \le c\varepsilon^2, \quad \forall\, t \in [0, T^\varepsilon).$$

This estimate shows (3.2)-(3.3) and Theorem 3.1 by standard arguments (see [22, 4]). It also justifies (3.8) by a bootstrap argument (see [38]). □

4. Uniform global existence and global convergence

In this section, we want to prove the following result.

Theorem 4.1. *(Uniform global existence and global convergence for the relaxed Euler system) Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Let $m > \frac{d}{2} + 1$ be an integer and $(\rho_0^\varepsilon - 1, u_0^\varepsilon, v_0^{I,\varepsilon}, v_0^{II,\varepsilon}) \in H^m$. There are two positive constants δ and c (independent of ε) such that if*

$$\|\rho_0^\varepsilon - 1\|_m + \|u_0^\varepsilon\|_m + \sqrt{\varepsilon_1}\,\|v_0^{I,\varepsilon}\|_m + \sqrt{\varepsilon_2}\,\|v_0^{II,\varepsilon}\|_m \le \delta, \tag{4.1}$$

then for all $\varepsilon_1, \varepsilon_2 \in (0,1]$, the Cauchy problem (2.10) admits a unique global solution $(\rho^\varepsilon, u^\varepsilon, v^{I,\varepsilon}, v^{II,\varepsilon})$ satisfying

$$\begin{aligned} &\|\rho^\varepsilon(t) - 1\|_m^2 + \|u^\varepsilon(t)\|_m^2 + \varepsilon_1\,\|v^{I,\varepsilon}(t)\|_m^2 + \varepsilon_2\,\|v^{II,\varepsilon}(t)\|_m^2 \\ &+ \int_0^t \big(\|\nabla\rho^\varepsilon(t')\|_{m-1}^2 + \|\nabla u^\varepsilon(t')\|_{m-1}^2 + \|v^\varepsilon(t')\|_m^2\big) dt' \\ \le\ & c\big(\|\rho_0^\varepsilon - 1\|_m^2 + \|u_0^\varepsilon\|_m^2 + \varepsilon_1\|v_0^{I,\varepsilon}\|_m^2 + \varepsilon_2\|v_0^{II,\varepsilon}\|_m^2\big), \quad \forall\, t \ge 0. \end{aligned} \tag{4.2}$$

Moreover, there exist functions (ρ, u, v) with $(\rho - 1, u) \in L^\infty(\mathbb{R}^+; H^m)$ and $v \in L^2(\mathbb{R}^+; H^m)$, such that, as $\varepsilon \to 0$ and up to subsequences,

$$(\rho^\varepsilon, u^\varepsilon) \longrightarrow (\rho, u), \quad \textit{weakly-}* \ \textit{in } L^\infty(\mathbb{R}^+; H^m), \tag{4.3}$$

$$v^\varepsilon \longrightarrow \bar{v}, \quad \textit{weakly in } L^2(\mathbb{R}^+; H^m), \tag{4.4}$$

where

$$\bar{v} = -\sum_{j=1}^{d} N_j^T \partial_{x_j} u, \tag{4.5}$$

and (ρ, u) *is a unique solution to* (2.11) *for the compressible Navier-Stokes equations with initial value* (ρ_0, u_0) *being the weak limit of* $(\rho_0^\varepsilon, u_0^\varepsilon)$ *in* H^m *(up to subsequences).*

4.1. **Energy estimates for relaxed Euler equations.**

Recall that $\varepsilon = (\varepsilon_1, \varepsilon_2)$. According to [29], the global existence of smooth solutions follows from the local existence and uniform estimates of solutions with respect to t. The result will be uniform with respect to ε if the constants in the energy estimates are independent of ε. Since the local existence to (2.10) is known, it remains to establish uniform estimates with respect to t and ε. For simplicity, in this section the subscript ε in the expression of the solutions is dropped.

Let $T > 0$ be any time for which the smooth solution (ρ, u, v) to (2.10) is defined on time interval $[0, T]$,

$$(\rho - 1, u, v^I, v^{II}) \in C([0,T]; H^m) \cap C^1\big([0,T]; H^{m-1}\big).$$

In what follows, we denote

$$B_T = \sup_{0\le t\le T} \big(\|\rho(t) - 1\|_m + \|u(t)\|_m + \sqrt{\varepsilon_1}\|v^I(t)\|_m + \sqrt{\varepsilon_2}\|v^{II}(t)\|_m\big). \tag{4.6}$$

Since we consider smooth solutions near equilibrium state $(1, 0, 0)$ for (ρ, u, v), we may suppose that B_T is bounded by a sufficiently small constant independent of ε and T. Then $\frac{1}{2} \le \rho \le \frac{3}{2}$. It follows that $A_0(\rho)$ is uniformly positive definite with respect to ε.

The proof of Theorem 4.1 follows from an L^2 estimate, a higher order estimate and a dissipation estimate for $\nabla\rho$ and ∇u. Recall that in Lemmas 4.1-4.5 below, $c > 0$ and $c_i > 0\,(i \in \mathbb{N})$ are generic constants independent of ε, T and any time. Let us start with the L^2 estimate.

Lemma 4.1. *(L^2 estimate) For all* $\varepsilon \in (0,1]^2$ *and all* $t \in [0,T]$*, it holds*

$$\begin{aligned} &\|\rho(t) - 1\|^2 + \|u(t)\|^2 + \varepsilon_1\|v^I(t)\|^2 + \varepsilon_2\|v^{II}(t)\|^2 + \int_0^t \|v(t')\|^2 dt' \\ \le\ & c\Big(\|\rho_0 - 1\|^2 + \|u_0\|^2 + \varepsilon_1\|v_0^I\|^2 + \varepsilon_2\|v_0^{II}\|^2\Big). \end{aligned} \tag{4.7}$$

Proof. Let us recall the entropy equality in (2.9), which is

$$\partial_t E(U) + \operatorname{div} F(U) + |v|^2 = 0,$$

where E and F are defined in (2.8). By the Taylor formulae, there is a ρ_* between 1 and ρ such that

$$H(\rho) = H(1) + h(1)(\rho - 1) + h'(\rho_*)(\rho - 1)^2.$$

Using the density conservation in (2.10), we obtain

$$\partial_t\Big(\frac{1}{2}\rho|u|^2 + h'(\rho_*)(\rho-1)^2 + \frac{\varepsilon_1}{2}|v^I|^2 + \frac{\varepsilon_2}{2}|v^{II}|^2\Big) + \operatorname{div}\Big(F(U) - h(1)\rho u\Big) + |v|^2 = 0. \tag{4.8}$$

Since $h'(\rho) > 0$ for $\rho > 0$, when B_T is sufficiently small, we have

$$c_1\Big((\rho-1)^2 + |u|^2 + \varepsilon|v|^2\Big) \le \frac{1}{2}\rho|u|^2 + h'(\rho_*)(\rho-1)^2 + \frac{\varepsilon}{2}|v|^2 \le c\Big((\rho-1)^2 + |u|^2 + \varepsilon|v|^2\Big).$$

Integrating (4.8) over $[0,t] \times \mathbb{R}^d$ with $t \in [0,T]$, it yields (4.7). □

Now we consider higher order estimates of U.

Lemma 4.2. *(Higher order estimate) For all $\varepsilon \in (0,1]^2$ and all $t \in [0,T]$, it holds*

$$\begin{aligned} &\|\rho(t)-1\|_m^2 + \|u(t)\|_m^2 + \varepsilon_1\|v^I(t)\|_m^2 + \varepsilon_2\|v^{II}(t)\|_m^2 + \int_0^t \|v(t')\|_m^2 dt' \\ \leq\ & c\Big(\|\rho_0-1\|_m^2 + \|u_0\|_m^2 + \varepsilon_1\|v_0^I\|_m^2 + \varepsilon_2\|v_0^{II}\|_m^2\Big) \\ &+ c\int_0^t \big(\|\nabla\rho(t')\|_{m-1} + \|\nabla u(t')\|_{m-1}\big)\|\nabla U(t')\|_{m-1}^2 dt'. \end{aligned} \tag{4.9}$$

Proof. Let $1 \leq |\alpha| \leq m$. Applying ∂_x^α to (2.6), we get

$$D_1(\varepsilon)\partial_t(\partial_x^\alpha U) + \sum_{j=1}^d A_j(\rho,u)\partial_{x_j}(\partial_x^\alpha U) = S(\partial_x^\alpha v) - \sum_{j=1}^d J_j^\alpha,$$

where

$$J_j^\alpha = \partial_x^\alpha\big(A_j(\rho,u)\partial_{x_j}U\big) - A_j(\rho,u)\partial_x^\alpha(\partial_{x_j}U).$$

Similarly to the proof of Theorem 3.1, we obtain the classical energy equality :

$$\begin{aligned} \frac{d}{dt}\big\langle A_0(\rho)D_1(\varepsilon)\partial_x^\alpha U, \partial_x^\alpha U\big\rangle &= 2\big\langle A_0(\rho)\partial_x^\alpha U, S(\partial_x^\alpha v)\big\rangle - 2\sum_{j=1}^d \big\langle A_0(\rho)\partial_x^\alpha U, J_j^\alpha\big\rangle \\ &\quad + \big\langle \operatorname{div}\vec{A}(\rho,u)\partial_x^\alpha U, \partial_x^\alpha U\big\rangle, \quad \forall\, t\in[0,T], \end{aligned} \tag{4.10}$$

where $\operatorname{div}\vec{A}(\rho,u)$ is defined in (3.7).

Obviously,

$$\begin{aligned} &c_1\Big(\|\partial_x^\alpha\rho\|^2 + \|\partial_x^\alpha u\|^2 + \varepsilon_1\|\partial_x^\alpha v^I\|^2 + \varepsilon_2\|\partial_x^\alpha v^{II}\|^2\Big) \\ \leq\ & \big\langle A_0(\rho)D_1(\varepsilon)\partial_x^\alpha U, \partial_x^\alpha U\big\rangle \\ \leq\ & c\Big(\|\partial_x^\alpha\rho\|^2 + \|\partial_x^\alpha u\|^2 + \varepsilon_1\|\partial_x^\alpha v^I\|^2 + \varepsilon_2\|\partial_x^\alpha v^{II}\|^2\Big), \end{aligned} \tag{4.11}$$

and

$$2\big\langle A_0(\rho)\partial_x^\alpha U, S(\partial_x^\alpha v)\big\rangle = -2\|\partial_x^\alpha v\|^2. \tag{4.12}$$

Moreover,

$$\big\langle A_0(\rho)\partial_x^\alpha U, J_j^\alpha\big\rangle = K_j^\alpha + \big\langle \rho\partial_x^\alpha u, N_j\big(\partial_x^\alpha(\rho^{-1}\partial_{x_j}v) - \rho^{-1}\partial_x^\alpha\partial_{x_j}v\big)\big\rangle,$$

where K_j^α are crossed terms from the Euler equations defined by

$$\begin{aligned} K_j^\alpha &= \big\langle h'(\rho)\partial_x^\alpha\rho, \partial_x^\alpha(u_j\partial_{x_j}\rho) - u_j\partial_x^\alpha\partial_{x_j}\rho\big\rangle + \big\langle h'(\rho)\partial_x^\alpha\rho, \partial_x^\alpha(\rho\partial_{x_j}u_j) - \rho\partial_x^\alpha\partial_{x_j}u_j\big\rangle \\ &\quad + \big\langle \rho\partial_x^\alpha u_j, \partial_x^\alpha(h'(\rho)\partial_{x_j}\rho) - h'(\rho)\partial_x^\alpha\partial_{x_j}\rho\big\rangle + \big\langle \rho\partial_x^\alpha u, \partial_x^\alpha(u_j\partial_{x_j}u) - u_j\partial_x^\alpha\partial_{x_j}u\big\rangle. \end{aligned} \tag{4.13}$$

Applying the Moser-type calculus inequalities and the Young inequality, we have

$$\big|K_j^\alpha\big| \leq c\|\nabla u\|_{m-1}\big(\|\nabla\rho\|_{m-1}^2 + \|\nabla u\|_{m-1}^2\big), \tag{4.14}$$

$$\big|\big\langle A_0(\rho)\partial_x^\alpha U, J_j^\alpha\big\rangle\big| \leq c\big(\|\nabla\rho\|_{m-1} + \|\nabla u\|_{m-1}\big)\|\nabla U\|_{m-1}^2. \tag{4.15}$$

Let us denote

$$U_1 = \begin{pmatrix} \rho \\ u \end{pmatrix}, \quad \tilde{B}_j(U_1) = \begin{pmatrix} h'(\rho)u_j & p'(\rho)e_j^T \\ p'(\rho)e_j & \rho u_j I_d \end{pmatrix},$$

$$\operatorname{div}\vec{B}(U_1) = \operatorname{diag}(h''(\rho), I_d)\partial_t\rho + \sum_{j=1}^d \partial_{x_j}\tilde{B}_j(U_1).$$

Since each N_j is a constant matrix, from (3.7) and the definition of A_0 and $\tilde{A}_j$, it is easy to see that

$$\langle \operatorname{div} \vec{A}(\rho,u)\partial_x^\alpha U, \partial_x^\alpha U\rangle = \langle \operatorname{div} \vec{B}(U_1)\partial_x^\alpha U_1, \partial_x^\alpha U_1\rangle, \tag{4.16}$$

which is independent of v. Using $\partial_t\rho = -\operatorname{div}(\rho u)$, we obtain

$$\big|\big\langle \operatorname{div} \vec{A}(\rho,u)\partial_x^\alpha U, \partial_x^\alpha U\big\rangle\big| \le c\big(\|\nabla\rho\|_{m-1} + \|\nabla u\|_{m-1}\big)^3. \tag{4.17}$$

Adding (4.10) for all $1 \le |\alpha|$ and integrating the resulting inequality over $[0,t]$, from Lemma 4.1 and (4.11)-(4.17), we obtain (4.9). □

Now we consider dissipation estimates of $\nabla\rho$ and ∇u. For this purpose, we introduce

$$\tilde{u} = \sum_{j=1}^d N_j^T \partial_{x_j} u, \quad \tilde{v} = \sum_{j=1}^d N_j \partial_{x_j} v.$$

We first establish the following relations.

Lemma 4.3. *For all $\beta \in \mathbb{N}^d$, it holds*

$$\begin{aligned}
\|\partial_x^\beta \tilde{u}\|^2 &= \nu\|\partial_x^\beta(\nabla u)\|^2 + \mu\|\partial_x^\beta(\operatorname{div} u)\|^2,\\
\|\partial_x^\beta \tilde{v}\| &\le c\|\partial_x^\beta(\nabla v)\|,\\
\langle \partial_x^\beta v, \partial_x^\beta \tilde{u}\rangle &= -\langle \partial_x^\beta u, \partial_x^\beta \tilde{v}\rangle,\\
\langle D_0(\varepsilon)\partial_t\partial_x^\beta v, \partial_x^\beta \tilde{u}\rangle &= \frac{d}{dt}\langle D_0(\varepsilon)\partial_x^\beta v, \partial_x^\beta \tilde{u}\rangle + \sum_{j=1}^d \langle \partial_t\partial_x^\beta u, N_j D_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle.
\end{aligned}$$

Proof. Obviously, it suffices to prove the result for $\beta = 0$. By the definition, $N_j = (\sqrt{\nu}M_j, \sqrt{\mu}e_j)$. Since $e_j^T u = u_j$, we have

$$\langle N_i^T\partial_{x_i}u, N_j^T\partial_{x_j}u\rangle = -\nu\langle M_i M_j^T \partial_{x_ix_j}^2 u, u\rangle + \mu\langle \partial_{x_i}u_i, \partial_{x_j}u_j\rangle.$$

Hence, by (2.1),

$$\|\tilde{u}\|^2 = \nu\|\nabla u\|^2 + \mu\|\operatorname{div} u\|^2.$$

This proves the first equality in the lemma. The other three relations can be proved in a similar way. □

Lemma 4.4. *(Dissipation estimates of $\nabla\rho$ and ∇u) There are positive constants c_2, c_3 and c_4 such that for all $\varepsilon \in (0,1]^2$ and all $t \in [0,T]$, it holds*

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\beta|\le m-1}\big[\langle D_0(\varepsilon)\partial_x^\beta v, \partial_x^\beta \tilde{u}\rangle + c_2\langle \partial_x^\beta u, \partial_x^\beta(\nabla h(\rho))\rangle\big] + c_3\|\nabla\rho\|_{m-1}^2 + c_4\|\nabla u\|_{m-1}^2\\
&\le c\|v\|_m^2 + c\|u\|_m\big(\|\nabla\rho\|_{m-1}^2 + \|\nabla u\|_{m-1}^2\big).
\end{aligned} \tag{4.18}$$

Proof. Let $\beta \in \mathbb{N}^d$ with $|\beta| \le m-1$. Applying ∂_x^β to the system in (2.10), we have

$$\begin{cases}
\partial_t\partial_x^\beta\rho + \operatorname{div}\partial_x^\beta(\rho u) = 0,\\
\partial_t\partial_x^\beta u + \partial_x^\beta\big((u\cdot\nabla)u\big) + \partial_x^\beta(\nabla h(\rho)) + \partial_x^\beta\Big(\dfrac{1}{\rho}\tilde{v}\Big) = 0,\\
D_0(\varepsilon)\partial_t\partial_x^\beta v + \partial_x^\beta\tilde{u} = -\partial_x^\beta v.
\end{cases} \tag{4.19}$$

Taking the inner product of the third equation of (4.19) with $\partial_x^\beta \tilde{u}$ in $L^2(\mathbb{R}^d)$, by Lemma 4.3 we obtain

$$\begin{aligned}\nu\|\partial_x^\beta(\nabla u)\|^2+\mu\|\partial_x^\beta(\operatorname{div} u)\|^2 &= -\langle\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle-\langle D_0(\varepsilon)\partial_t\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle\\ &= -\langle\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle-\frac{d}{dt}\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle\\ &\quad-\sum_{j=1}^d\langle\partial_t\partial_x^\beta u,N_jD_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle.\end{aligned}$$

Since $\mu\geq 0$, by the Young inequality, the above equality implies that

$$2\frac{d}{dt}\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle+\nu\|\partial_x^\beta(\nabla u)\|^2\leq c\|v\|_m^2-2\sum_{j=1}^d\langle\partial_t\partial_x^\beta u,N_jD_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle. \tag{4.20}$$

For the last term in (4.20), we use the second equation in (4.19) to obtain

$$-\langle\partial_t\partial_x^\beta u,N_jD_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle=\langle\partial_x^\beta\big((u\cdot\nabla)u+\nabla h(\rho)+\rho^{-1}\tilde{v}\big),N_jD_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle.$$

Since $\varepsilon\in(0,1]^2$, by Lemma 4.3 and the Moser-type calculus inequalities, we have

$$-\sum_{j=1}^d\langle\partial_t\partial_x^\beta u,N_jD_0(\varepsilon)\partial_{x_j}\partial_x^\beta v\rangle\leq\eta\|\partial_x^\beta(\nabla h(\rho))\|^2+c\|v\|_m^2+c\|u\|_m\|\nabla u\|_{m-1}\|v\|_m,$$

where $\eta>0$ is a small constant to be chosen. This inequality together with (4.20) gives

$$2\frac{d}{dt}\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle+\nu\|\partial_x^\beta(\nabla u)\|^2\leq\eta\|\partial_x^\beta(\nabla h(\rho))\|^2+c\|v\|_m^2+c\|u\|_m\|\nabla u\|_{m-1}^2. \tag{4.21}$$

Next, taking the inner product of the second equation of (4.19) with $\partial_x^\beta(\nabla h(\rho))$ in $L^2(\mathbb{R}^d)$ yields

$$\|\partial_x^\beta(\nabla h(\rho))\|^2=-\langle\partial_t\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle-\langle\partial_x^\beta\big((u\cdot\nabla)u+\rho^{-1}\tilde{v}\big),\partial_x^\beta(\nabla h(\rho))\rangle.$$

Hence, by the Young inequality,

$$\|\partial_x^\beta(\nabla h(\rho))\|^2\leq c\|\nabla u\|_{m-1}^2+c\|v\|_m^2-2\langle\partial_t\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle. \tag{4.22}$$

For the last term in (4.22), we use the equation of the density conservation

$$\partial_t\rho=-\operatorname{div}(\rho u).$$

It follows from an integration by parts that

$$-2\langle\partial_t\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle=-2\frac{d}{dt}\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle+2\langle\partial_x^\beta\operatorname{div}u,\partial_x^\beta(h'(\rho)\operatorname{div}(\rho u))\rangle.$$

Therefore, by the Young inequality and the Moser-type calculus inequalities,

$$-2\langle\partial_t\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle\leq-2\frac{d}{dt}\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle+c\|\nabla u\|_{m-1}^2+c\|\nabla\rho\|_{m-1}^2\|u\|_m.$$

This inequality together with (4.22) gives

$$2\frac{d}{dt}\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle+\|\partial_x^\beta(\nabla h(\rho))\|^2\leq c\|\nabla u\|_{m-1}^2+c\|v\|_m^2+c\|\nabla\rho\|_{m-1}^2\|u\|_m. \tag{4.23}$$

Multiplying (4.23) by 2η and adding with (4.21) yields

$$\begin{aligned}&\frac{d}{dt}\big[2\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle+4\eta\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle\big]+\eta\|\partial_x^\beta(\nabla h(\rho))\|^2+\nu\|\partial_x^\beta(\nabla u)\|^2\\ \leq\;& c\eta\|\nabla u\|_{m-1}^2+c\|v\|_m^2+c\|u\|_m\big(\|\nabla\rho\|_{m-1}^2+\|\nabla u\|_{m-1}^2\big).\end{aligned}$$

Adding these inequalities for all β and taking η small enough, since

$$\sum_{|\beta|\le m-1}\|\partial_x^\beta(\nabla u)\|^2=\|\nabla u\|_{m-1}^2,$$

we have

$$\begin{aligned}&\frac{d}{dt}\sum_{|\beta|\le m-1}\big[2\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle+4\eta\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle\big]+\eta\|\nabla h(\rho)\|_{m-1}^2+\frac{\nu}{2}\|\nabla u\|_{m-1}^2\\ \le\;& c\|v\|_m^2+c\|u\|_m\big(\|\nabla\rho\|_{m-1}^2+\|\nabla u\|_{m-1}^2\big).\end{aligned}$$

Finally, recall that h is defined by $h'(\rho)=p'(\rho)/\rho$. Since p is sufficiently smooth, so is h (say $h\in C^m(\mathbb{R}_*^+)$). By the Moser-type calculus inequalities (see Proposition 2.1 (C) in [22], p.43), we have

$$\|\nabla h(\rho)\|_{m-1}\le c\|\nabla\rho\|_{m-1}.$$

On the other hand, $h'(1)=p'(1)>0$ implies that h is a C^m-diffeomorphism at least in a neighborhood of $\rho=1$. Thus, $\|\nabla h(\rho)\|_{m-1}$ is uniformly equivalent to $\|\nabla\rho\|_{m-1}$. This proves (4.18). □

Lemma 4.5. *(Final estimate) For all $\varepsilon\in(0,1]^2$ and all $t\in[0,T]$, it holds*

$$\begin{aligned}&\|\rho(t)-1\|_m^2+\|u(t)\|_m^2+\varepsilon_1\|v^I(t)\|_m^2+\varepsilon_2\|v^{II}(t)\|_m^2\\ &+\int_0^t\big(\|\nabla\rho(t')\|_{m-1}^2+\|\nabla u(t')\|_{m-1}^2+\|v(t')\|_m^2\big)dt'\\ \le\;& c\big(\|\rho_0-1\|_m^2+\|u_0\|_m^2+\varepsilon_1\|v_0^I\|_m^2+\varepsilon_2\|v_0^{II}\|_m^2\big)+cB_T\int_0^t\|\nabla U(t')\|_{m-1}^2dt',\end{aligned}\tag{4.24}$$

where B_T is defined in (4.6).

Proof. Notice that, for all $\beta\in\mathbb{N}^d$ with $|\beta|\le m-1$,

$$\big|\langle D_0(\varepsilon)\partial_x^\beta v,\partial_x^\beta\tilde{u}\rangle+c_2\langle\partial_x^\beta u,\partial_x^\beta(\nabla h(\rho))\rangle\big|\le c\big(\|\nabla\rho\|_{m-1}^2+\|u\|_m^2+\varepsilon_1\|v^I\|_m^2+\varepsilon_2\|v^{II}\|_m^2\big).$$

Integrating (4.18) over $[0,t]$ and combining the result with (4.9), we obtain (4.24). □

4.2. **Proof of Theorem 4.1.**

When the solution U is uniformly small in $L^\infty(0,T;H^m)$, the integral on the right hand-side can be controlled by that of the left-side in (4.24). This implies the uniform estimate (4.2) and the uniform global existence result.

Now we prove the global convergence. The uniform estimate (4.2) implies that the sequences $(\rho^\varepsilon-1)_{\varepsilon>0}$ and $(u^\varepsilon)_{\varepsilon>0}$ are bounded in $L^\infty(\mathbb{R}^+;H^m)$ and the sequence $(v^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\mathbb{R}^+;H^m)$. This ensures the convergence (4.3)-(4.4) and

$$D_0(\varepsilon)v^\varepsilon\longrightarrow 0,\quad\text{strongly in } L^2(\mathbb{R}^+;H^m).$$

Since $(\rho^\varepsilon,u^\varepsilon,v^\varepsilon)$ is a solution to (2.5), we have

$$\begin{cases}\partial_t\rho^\varepsilon+\operatorname{div}(\rho^\varepsilon u^\varepsilon)=0,\\ \partial_t(\rho^\varepsilon u^\varepsilon)+\operatorname{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon)+\nabla p(\rho^\varepsilon)+\displaystyle\sum_{j=1}^d N_j\partial_{x_j}v^\varepsilon=0,\\ D_0(\varepsilon)\partial_t v^\varepsilon+\displaystyle\sum_{j=1}^d N_j^T\partial_{x_j}u^\varepsilon=-v^\varepsilon.\end{cases}\tag{4.25}$$

For all $T>0$, it is easy to see that both $(\partial_t\rho^\varepsilon)_{\varepsilon>0}$ and $(\partial_t u^\varepsilon)_{\varepsilon>0}$ are bounded in $L^2(0,T;H^{m-1})$. Hence, $(\rho,u)\in C([0,T];H^{m-1})$. Moreover, by a classical compactness theorem (see [37]), for

all $m_1 \in (0, m)$, $(\rho^\varepsilon)_{\varepsilon>0}$ and $(u^\varepsilon)_{\varepsilon>0}$ are relatively compact in $C([0,T]; H^{m_1}_{loc})$. As a consequence, as $\varepsilon \to 0$ and up to subsequences,

$$(\rho^\varepsilon, u^\varepsilon) \longrightarrow (\rho, u), \quad \text{strongly in } C([0,T]; H^{m_1}_{loc}).$$

Passing to the limit in the sense of distributions in (4.25), we obtain the Navier-Stokes equations (2.11) and also (4.5).

Now we describe the initial condition for (ρ, u). The convergence above is uniform with respect to time. Hence,

$$(\rho^\varepsilon_0, u^\varepsilon_0) = (\rho^\varepsilon(0,\cdot), u^\varepsilon(0,\cdot)) \longrightarrow (\rho(0,\cdot), u(0,\cdot)), \quad \text{strongly in } H^{m_1}_{loc}.$$

Moreover, $(\rho, u) \in C([0,T]; H^{m-1})$ implies that $(\rho(0,\cdot), u(0,\cdot)) \in H^{m-1}$. On the other hand, the boundedness of $(\rho^\varepsilon_0 - 1, u^\varepsilon_0)$ in H^m implies that, up to a subsequence, $(\rho^\varepsilon_0, u^\varepsilon_0)$ admits a weak limit in H^m, denoted by $(\rho_0, u_0) \in H^m$. It follows that, for all $R > 0$

$$(\rho(0,\cdot), u(0,\cdot)) = (\rho_0, u_0), \quad \text{in } B_R,$$

where B_R is the ball of radius R and center 0 in $\mathbb{R}^d$. Now $m - 1 > d/2$ and the embedding $H^{m-1}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$ is continuous. Hence, both $(\rho(0,\cdot), u(0,\cdot))$ and (ρ_0, u_0) are continuous functions in $\mathbb{R}^d$. Since $R > 0$ is arbitrary, we conclude that

$$(\rho(0,\cdot), u(0,\cdot)) = (\rho_0, u_0), \quad \text{in } \mathbb{R}^d.$$

This ends the proof of Theorem 4.1.

5. Incompressible relaxed Euler systems

Let $d \geq 2$ and $\nu > 0$. Consider incompressible Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u, \tag{5.1}$$

$$\operatorname{div} u = 0, \tag{5.2}$$

in $\mathbb{R}^+ \times \mathbb{R}^d$, where $p(t,x)$ vanishes sufficiently fast as $|x| \to +\infty$. Using

$$\operatorname{div}\big((u \cdot \nabla)u\big) = (u \cdot \nabla) \operatorname{div} u + \operatorname{tr}\big((\nabla u)^2\big), \quad \operatorname{tr}\big((\nabla u)^2\big) = \sum_{i,j=1}^{d} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}, \tag{5.3}$$

it is known that p satisfies a Poisson equation

$$-\Delta p = \operatorname{tr}\big((\nabla u)^2\big), \tag{5.4}$$

whose solution is given by

$$p(t,x) = \int_{\mathbb{R}^d} G(x-y) \operatorname{tr}\big((\nabla u(t,y))^2\big) dy,$$

where G is the fundamental solution of the Laplace equation :

$$G(x) = \begin{cases} -\dfrac{1}{2\pi} \ln|x|, & \text{if } d = 2, \\ \dfrac{C_d}{|x|^{d-2}}, & \text{if } d \geq 3, \end{cases}$$

with $C_d > 0$ being a constant.

From the discussion in the compressible case, we propose the relaxed incompressible Euler systems as follows

$$\operatorname{div} u = 0, \tag{5.5}$$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p + \sqrt{\nu}\sum_{j=1}^{d} M_j \partial_{x_j} v = 0, \\ \varepsilon\partial_t v + \sqrt{\nu}\sum_{j=1}^{d} M_j^T \partial_{x_j} u = -v, \end{cases} \tag{5.6}$$

in $\mathbb{R}^+\times\mathbb{R}^d$, where $v\in\mathbb{R}^r$. Consider the Cauchy problem for (5.5)-(5.6) with initial condition

$$t=0 : (u,v) = (u_0^\varepsilon, v_0^\varepsilon). \tag{5.7}$$

Suppose u_0^ε and v_0^ε are smooth and

$$\operatorname{div} u_0^\varepsilon = 0, \quad \operatorname{div}\left(\sum_{j=1}^{d} M_j\partial_{x_j} v_0^\varepsilon\right) = 0. \tag{5.8}$$

For incompressible Navier-Stokes equations, suppose (5.1) and $\operatorname{div} u(0,x) = 0$ hold. It is known that (see [23]) the incompressibility condition (5.5) is equivalent to (5.4) for all $t>0$. Now we establish a similar result for the relaxed Euler systems.

Proposition 5.1. *Let $T>0$ and $(u_0^\varepsilon, v_0^\varepsilon)\in H^2$ satisfying (5.8). Let (u,p,v) be a solution of (5.6)-(5.7) with regularity*

$$(u,v)\in L^\infty(0,T;H^2)\cap W^{1,\infty}(0,T;H^1), \quad \nabla u\in L^\infty\big((0,T)\times\mathbb{R}^d\big), \quad p\in L^\infty(0,T;H^2).$$

Then (5.5) is equivalent to (5.4) in $(0,T)\times\mathbb{R}^d$. As a consequence, we also have

$$\operatorname{div}\left(\sum_{j=1}^{d} M_j\partial_{x_j} v(t,\cdot)\right) = 0, \quad \forall\, t\in[0,T]. \tag{5.9}$$

Proof. As in Section 4, we denote $\tilde v = \sum_{j=1}^{d} M_j\partial_{x_j} v$. Multiplying the second equation in (5.6) on the left by M_i and summing up for all $i=1,\cdots,d$, by (2.2), we have

$$\varepsilon\partial_t\tilde v + \sqrt{\nu}\Delta u = -\tilde v.$$

Applying div to the above equation and to the first equation in (5.6), by (5.3), we have

$$\begin{cases} \partial_t(\operatorname{div} u) + u\cdot\nabla\operatorname{div} u + \big(\operatorname{tr}\big((\nabla u)^2\big) + \Delta p\big) + \sqrt{\nu}\operatorname{div}\tilde v = 0, \\ \varepsilon\partial_t(\operatorname{div}\tilde v) + \sqrt{\nu}\Delta(\operatorname{div} u) + \operatorname{div}\tilde v = 0. \end{cases}$$

If (5.5) holds, then

$$\begin{cases} \big(\operatorname{tr}\big((\nabla u)^2\big) + \Delta p\big) + \sqrt{\nu}\operatorname{div}\tilde v = 0, \\ \varepsilon\partial_t(\operatorname{div}\tilde v) + \operatorname{div}\tilde v = 0. \end{cases}$$

Obviously, the second equation in the above system together with the second condition in (5.8) yields $\operatorname{div}\tilde v = 0$ which is (5.9). This implies (5.4).

Conversely, if (5.4) holds, then

$$\begin{cases} \partial_t u_D + u\cdot\nabla u_D + \sqrt{\nu} v_D = 0, \\ \varepsilon\partial_t v_D + \sqrt{\nu}\Delta u_D + v_D = 0, \end{cases} \tag{5.10}$$

where $(u_D, v_D) = (\operatorname{div} u, \operatorname{div}\tilde v)$. A standard energy estimate for v_D yields

$$\frac{\varepsilon}{2}\frac{d}{dt}\|v_D\|^2 + \|v_D\|^2 = \sqrt{\nu}\,\langle\nabla v_D, \nabla u_D\rangle.$$

Since

$$\sqrt{\nu} v_D = -\partial_t u_D - u \cdot \nabla u_D,$$

we have

$$\sqrt{\nu}\, \langle \nabla v_D, \nabla u_D \rangle = -\frac{1}{2}\frac{d}{dt}\|\nabla u_D\|^2 - \langle \nabla(u \cdot \nabla u_D), \nabla u_D \rangle.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon \|v_D\|^2 + \|\nabla u_D\|^2\big) + \|v_D\|^2 = -\langle \nabla(u \cdot \nabla u_D), \nabla u_D \rangle.$$

A straightforward calculation shows that

$$\langle \nabla(u \cdot \nabla u_D), \nabla u_D \rangle = \sum_{j=1}^{d} \langle \nabla u_j \cdot \partial_{x_j} u_D, \nabla u_D \rangle - \langle u_D, |\nabla u_D|^2 \rangle.$$

Then, by denoting $C_T = \|\nabla u\|_{L^\infty((0,T)\times\mathbb{R}^d)}$, we have

$$\big|\langle \nabla(u \cdot \nabla u_D), \nabla u_D \rangle\big| \le C_T \|\nabla u_D\|^2,$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon \|v_D\|^2 + \|\nabla u_D\|^2\big) + \|v_D\|^2 \le C_T \|\nabla u_D\|^2.$$

By the Gronwall inequality together with (5.8), we obtain

$$v_D(t,\cdot) = 0, \quad \nabla u_D(t,\cdot) = 0, \quad \forall\, t \in (0,T).$$

By (5.10), we further obtain $\partial_t u_D(t,\cdot) = 0$ for all $t \in (0,T)$, which implies (5.5) and (5.9). □

Let $m > \dfrac{d}{2} + 1$ be an integer. Suppose $u_0^\varepsilon, v_0^\varepsilon \in H^m$. Similarly to the energy estimates in Sections 4-5, it is easy to see that the solution of (5.5)-(5.7) satisfies

$$\frac{d}{dt}\big(\|u\|_m^2 + \varepsilon \|v\|_m^2\big) + \|v\|_m^2 \le C\|u\|_m^3,$$

where $C > 0$ is independent of (u, v). Thus, we may adapt the proof of existence of solutions for incompressible Euler equations [20, 16, 23]. There exist a maximal time $T_0^\varepsilon > 0$ and a unique smooth solution (u, p, v) to the Cauchy problem (5.5)-(5.7) such that

$$u, v \in C([0, T_0^\varepsilon); H^m) \cap C^1\big([0, T_0^\varepsilon); H^{m-1}\big), \quad \nabla p \in L^2\big((0, T_0^\varepsilon); H^{m-1}\big).$$

Theorem 5.1. *(Local convergence)*

Let $m > \dfrac{d}{2} + 1$ be an integer. Let $(u_0^\varepsilon, v_0^\varepsilon) \in H^m$, $u_0 \in H^{m+2}$ and $v_0 \in H^{m+1}$. We assume (5.8) *holds and*

$$\|u_0^\varepsilon - u_0\|_m + \sqrt{\varepsilon}\, \|v_0^\varepsilon - v_0\|_m \le c_1 \varepsilon, \tag{5.11}$$

where $c_1 > 0$ is a constant independent of ε.

Let (u, p) be the unique solution on $[0, T_0]$ to (5.1)-(5.2) *with initial data u_0. Then there exists a constant $\varepsilon_0 \in (0, 1]$ depending on T_0 such that for all $\varepsilon \in (0, \varepsilon_0]$, the unique solution $(u^\varepsilon, p^\varepsilon, v^\varepsilon)$ to* (5.5)-(5.7) *is defined on $[0, T_0]$, and we have*

$$\|u^\varepsilon(t) - u(t)\|_m^2 + \int_0^t \|v^\varepsilon(t') - \bar{v}(t') - v_\varepsilon(t')\|_m^2 dt' \le c\varepsilon^2, \quad \forall\, t \in [0, T_0], \tag{5.12}$$

$$\|v^\varepsilon(t) - \bar{v}(t) - v_\varepsilon(t)\|_m \le c\sqrt{\varepsilon}, \quad \forall\, t \in [0, T_0], \tag{5.13}$$

$$\int_0^t \|\nabla(p^\varepsilon(t') - p(t'))\|_{m-1}^2 dt' \le c\varepsilon, \quad \forall\, t \in [0, T_0], \tag{5.14}$$

where $\bar{v}$ and v_ε are defined in (2.12)-(2.13)*, and $c > 0$ is a constant independent of ε.*

Proof. Let us introduce

$$w^\varepsilon = u^\varepsilon - u, \quad q^\varepsilon = p^\varepsilon - p, \quad z^\varepsilon = v^\varepsilon - (\bar{v} + v_\varepsilon),$$
$$w_0^\varepsilon = u_0^\varepsilon - u_0, \quad z_0^\varepsilon = v_0^\varepsilon - v_0.$$

From (5.1)-(5.4), we have

$$\begin{cases} \partial_t w^\varepsilon + (u^\varepsilon \cdot \nabla) w^\varepsilon + \nabla q^\varepsilon + \sqrt{\nu} \sum_{j=1}^d M_j \partial_{x_j} z^\varepsilon = -(w^\varepsilon \cdot \nabla) u - \sqrt{\nu} \sum_{j=1}^d M_j \partial_{x_j} v_\varepsilon, \\ \varepsilon \partial_t z^\varepsilon + \sqrt{\nu} \sum_{j=1}^d M_j^T \partial_{x_j} w^\varepsilon = -z^\varepsilon - \varepsilon \partial_t \bar{v}, \quad (t,x) \in [0, T^\varepsilon) \times \mathbb{R}^d, \end{cases} \tag{5.15}$$

where $T^\varepsilon = \min(T_0^\varepsilon, T_0) \in (0, T_0]$. For $\alpha \in \mathbb{N}^d$ with $|\alpha| \le m$, applying ∂_x^α to (5.15) yields

$$\begin{cases} \partial_t \partial_x^\alpha w^\varepsilon + (u^\varepsilon \cdot \nabla) \partial_x^\alpha w^\varepsilon + \nabla \partial_x^\alpha q^\varepsilon + \sqrt{\nu} \sum_{j=1}^d M_j \partial_{x_j} \partial_x^\alpha z^\varepsilon = L_\alpha^\varepsilon - \sqrt{\nu} \sum_{j=1}^d M_j \partial_{x_j} \partial_x^\alpha v_\varepsilon, \\ \varepsilon \partial_t \partial_x^\alpha z^\varepsilon + \sqrt{\nu} \sum_{j=1}^d M_j^T \partial_x^\alpha \partial_{x_j} w^\varepsilon = -\partial_x^\alpha z^\varepsilon - \varepsilon \partial_x^\alpha \partial_t \bar{v}, \quad (t,x) \in [0, T^\varepsilon) \times \mathbb{R}^d, \end{cases}$$

where

$$L_\alpha^\varepsilon = \big[(u^\varepsilon \cdot \nabla) \partial_x^\alpha w^\varepsilon - \partial_x^\alpha \big((u^\varepsilon \cdot \nabla) w^\varepsilon\big)\big] - \partial_x^\alpha \big((w^\varepsilon \cdot \nabla) u\big).$$

Since $\operatorname{div} u^\varepsilon = \operatorname{div} w^\varepsilon = 0$, we have

$$\langle \partial_x^\alpha w^\varepsilon, \nabla \partial_x^\alpha q^\varepsilon \rangle = 0,$$
$$\langle \partial_x^\alpha w^\varepsilon, (u^\varepsilon \cdot \nabla) \partial_x^\alpha w^\varepsilon \rangle = 0.$$

We also have

$$\langle \partial_x^\alpha w^\varepsilon, M_j \partial_{x_j}^\alpha \partial_x^\alpha z^\varepsilon \rangle = -\langle \partial_x^\alpha z^\varepsilon, M_j^T \partial_{x_j}^\alpha \partial_x^\alpha w^\varepsilon \rangle.$$

Hence, a classical energy estimate yields

$$\begin{aligned} &\frac{d}{dt}\big(\|\partial_x^\alpha w^\varepsilon\|^2 + \varepsilon \|\partial_x^\alpha z^\varepsilon\|^2\big) + 2\|\partial_x^\alpha z^\varepsilon\|^2 \\ &= 2\langle \partial_x^\alpha w^\varepsilon, L_\alpha^\varepsilon \rangle - 2\sqrt{\nu} \sum_{j=1}^d \langle \partial_x^\alpha w^\varepsilon, M_j \partial_{x_j} \partial_x^\alpha v_\varepsilon \rangle - 2\varepsilon \langle \partial_x^\alpha z^\varepsilon, \partial_x^\alpha \partial_t \bar{v} \rangle. \end{aligned} \tag{5.16}$$

By the Moser-type calculus inequalities, we have

$$2\big|\langle \partial_x^\alpha w^\varepsilon, L_\alpha^\varepsilon \rangle\big| \le c \|w^\varepsilon\|_m^2.$$

In a similar way to (3.11) and (3.12), we estimate the last two terms in (5.16) and obtain

$$2\sqrt{\nu}\big|\langle \partial_x^\alpha w^\varepsilon, M_j \partial_{x_j} \partial_x^\alpha v_\varepsilon \rangle\big| \le c \|w^\varepsilon\|_m e^{-\frac{t}{\varepsilon}},$$
$$2\varepsilon\big|\langle \partial_x^\alpha z^\varepsilon, \partial_x^\alpha \partial_t \bar{v} \rangle\big| \le \|\partial_x^\alpha z^\varepsilon\|^2 + c\varepsilon^2.$$

Taking into account these estimates, we add (5.16) for all α to get

$$\frac{d}{dt}\big(\|w^\varepsilon\|_m^2 + \varepsilon \|z^\varepsilon\|_m^2\big) + \|z^\varepsilon\|_m^2 \le c\|w^\varepsilon\|_m^2 + c\|w^\varepsilon\|_m e^{-\frac{t}{\varepsilon}} + c\varepsilon^2.$$

Thus, a Gronwall inequality together with condition (5.11) implies that

$$\|w^\varepsilon(t)\|_m^2 + \varepsilon \|z^\varepsilon(t)\|_m^2 + \int_0^t \|z^\varepsilon(t')\|_m^2 ds \le c\varepsilon^2.$$

This proves (5.12)-(5.13).

Finally, noting $z^\varepsilon + v_\varepsilon = v^\varepsilon - \bar{v}$, taking the divergence of the first equation in (5.15), we have

$$-\Delta q^\varepsilon = \operatorname{div}\Big((u^\varepsilon\cdot\nabla)w^\varepsilon + (w^\varepsilon\cdot\nabla)u + \sqrt{\nu}\sum_{j=1}^{d} M_j\partial_{x_j}(v^\varepsilon - \bar{v})\Big).$$

By (5.12), this implies (5.14). The proof of Theorem 5.1 is finished. □

As a variant of Theorem 4.1, we also have a result on the uniform global existence and global convergence for relaxed incompressible Euler equations. The proof is omitted here.

Theorem 5.2. *(Uniform global existence and global convergence)*

Let $m > \dfrac{d}{2} + 1$ be an integer. We assume $(u_0^\varepsilon, v_0^\varepsilon) \in H^m$ and (5.8) holds. There are two positive constants δ and c (independent of ε) such that if

$$\|u_0^\varepsilon\|_m + \sqrt{\varepsilon}\,\|v_0^\varepsilon\|_m \le \delta, \tag{5.17}$$

then for all $\varepsilon \in (0,1]$, the Cauchy problem (5.5)-(5.7) admits a unique global solution $(u^\varepsilon, p^\varepsilon, v^\varepsilon)$ satisfying

$$\begin{aligned}\|u^\varepsilon(t)\|_m^2 + \varepsilon\,\|v^\varepsilon(t)\|_m^2 + \int_0^t \big(\|\nabla u^\varepsilon(t')\|_{m-1}^2 + \|\nabla p^\varepsilon(t')\|_{m-1}^2 + \|v^\varepsilon(t')\|_m^2\big)dt' \\ \le\ c\big(\|u_0^\varepsilon\|_m^2 + \varepsilon\|v_0^\varepsilon\|_m^2\big), \quad \forall\, t \ge 0.\end{aligned} \tag{5.18}$$

Moreover, there exist functions $u \in L^\infty(\mathbb{R}^+; H^m)$ and p, $\bar{v} \in L^2(\mathbb{R}^+; H^m)$, such that, as $\varepsilon \to 0$ and up to subsequences,

$$u^\varepsilon \longrightarrow u, \quad \textit{weakly-}* \ \textit{in } L^\infty(\mathbb{R}^+; H^m). \tag{5.19}$$

$$\nabla p^\varepsilon \longrightarrow \nabla p, \quad v^\varepsilon \longrightarrow \bar{v}, \quad \textit{weakly in } L^2(\mathbb{R}^+; H^m), \tag{5.20}$$

where

$$\bar{v} = -\sqrt{\nu}\sum_{j=1}^{d} M_j^T\partial_{x_j}u, \tag{5.21}$$

and (u,p) is a unique solution to the Cauchy problem for incompressible Navier-Stokes equations (5.1)-(5.2) with initial value u_0 being the weak limit of u_0^ε in H^m (up to subsequences).

6. Relaxed Euler systems with tensor variables

6.1. The system with Maxwell's constitutive relation.

We consider system (1.5), namely,

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0,\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) + \operatorname{div}\pi = 0,\\ \varepsilon\partial_t\pi + \nu\sigma(u) + \lambda(\operatorname{div}u)I_d = -\pi, \quad \text{in } \mathbb{R}^+\times\mathbb{R}^d,\end{cases} \tag{6.1}$$

with initial condition

$$t = 0: \quad (\rho, u, \pi) = (\rho_0^\varepsilon, u_0^\varepsilon, \pi_0^\varepsilon), \tag{6.2}$$

where π is a symmetric matrix variable of order d and $\varepsilon > 0$ is a small parameter.

Recall the inner product of two matrices as follows :

$$\pi : \tau = \sum_{i,j=1}^{d}\pi_{ij}\tau_{ij}, \quad \text{for } \pi = (\pi_{ij})_{1\le i,j\le d},\ \ \tau = (\tau_{ij})_{1\le i,j\le d}.$$

We denote

$$|\pi|^2 = \pi : \pi = \sum_{i,j=1}^{d} \pi_{ij}^2,$$

and

$$\langle \pi, \tau \rangle = \int_{\mathbb{R}^d} \pi : \tau dx, \quad \|\pi\|^2 = \langle \pi, \pi \rangle, \quad \|\pi\|_m^2 = \sum_{|\alpha| \le m} \|\partial_x^\alpha \pi\|^2.$$

Obviously,

$$\operatorname{div}(\pi^T u) = u \cdot \operatorname{div} \pi + \pi : \nabla u, \quad \pi : I_d = \operatorname{tr}(\pi).$$

Now we consider the pair of entropy-entropy flux. Let (E_0, F_0) be the pair of entropy-entropy flux for the Euler equations, defined by (2.7). By (6.1), we have

$$\partial_t E_0(\rho, u) + \operatorname{div} F_0(\rho, u) + u \cdot \operatorname{div} \pi = 0.$$

By the symmetry of π and the definition of $\sigma(u)$, we have

$$\frac{1}{2}\pi : \sigma(u) = \pi : \nabla u - \frac{1}{d}(\operatorname{div} u)\operatorname{tr}(\pi).$$

Hence, the third equation in (6.1) yields an energy equality

$$\frac{\varepsilon}{4\nu}\partial_t |\pi|^2 + \frac{1}{2\nu}|\pi|^2 + \pi : \nabla u + \Big(\frac{\lambda}{2\nu} - \frac{1}{d}\Big)(\operatorname{div} u)\operatorname{tr}(\pi) = 0.$$

In order to eliminate the term containing $\operatorname{tr}(\pi)$, we take the trace of the third equation in (6.1). Since $\operatorname{tr}(\sigma(u)) = 0$, it yields

$$\varepsilon \partial_t \operatorname{tr}(\pi) + d\lambda \operatorname{div} u = -\operatorname{tr}(\pi),$$

which implies that

$$\frac{\varepsilon}{2}\partial_t (\operatorname{tr}(\pi))^2 + (\operatorname{tr}(\pi))^2 + d\lambda (\operatorname{div} u)\operatorname{tr}(\pi) = 0.$$

It follows from these two energy equalities that

$$\varepsilon \partial_t P_1(\pi) + 2P_1(\pi) + \pi : \nabla u = 0.$$

where

$$P_1(\pi) = \frac{1}{4\nu}\big(|\pi|^2 + \omega(\operatorname{tr}(\pi))^2\big), \quad \text{with } \omega = \frac{2\nu}{d^2\lambda} - \frac{1}{d}. \tag{6.3}$$

Let $V_1 = (\rho, u, \pi)$. We define functions E_1 and F_1 by

$$\begin{cases} E_1(V_1) = E_0(\rho, u) + \varepsilon P_1(\pi), \\ F_1(V_1) = F_0(\rho, u) + \pi u. \end{cases} \tag{6.4}$$

It is easy to check that a smooth solution V_1 of (6.1) satisfies the energy equality

$$\partial_t E_1(V_1) + \operatorname{div} F_1(V_1) + 2P_1(\pi) = 0. \tag{6.5}$$

Therefore, (E, F) is a pair of entropy-entropy flux of system (6.1). Moreover, if $\omega \ge 0$, then

$$P_1(\pi) \ge \frac{1}{4\nu}|\pi|^2,$$

and if $\omega < 0$, by using

$$(\operatorname{tr}(\pi))^2 \le d|\pi|^2,$$

we have

$$P_1(\pi) \ge \frac{1}{4\nu}(1 + \omega d)|\pi|^2 = \frac{1}{2d\lambda}|\pi|^2.$$

Hence,

$$P_1(\pi) \ge \min\Big(\frac{1}{4\nu}, \frac{1}{2d\lambda}\Big)|\pi|^2. \tag{6.6}$$

Since $\pi \longrightarrow P_1(\pi)$ is quadratic, E_1 is a strictly convex function with respect to the conservative variable $(\rho, \rho u, \pi)$ for $\rho > 0$. As a consequence, E_1 is a strictly convex entropy. If we use the columns of π as variables instead of π, system (6.1) can be written in a standard form. We conclude that system (6.1) is symmetrizable hyperbolic (see [12, 11, 2]) and its Cauchy problem admits a unique smooth solution defined in a finite time interval (see [19, 17, 22]).

The local convergence of system (6.1) can be proved in a similar way to the proof of Theorem 3.1. Now we establish the global convergence of the system near a constant state $(1, 0, 0)$ for V_1. The result is stated as follows.

Theorem 6.1. *Let $m > \frac{d}{2} + 1$ be an integer. Assume $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \pi_0^\varepsilon) \in H^m$ with π_0^ε being symmetric. There are two positive constants δ and c (independent of ε) such that if*

$$\|\rho_0^\varepsilon - 1\|_m + \|u_0^\varepsilon\|_m + \sqrt{\varepsilon}\,\|\pi_0^\varepsilon\|_m \le \delta,$$

then for all $\varepsilon \in (0, 1]$, the Cauchy problem (6.1)-(6.2) admits a unique global solution $(\rho^\varepsilon, u^\varepsilon, \pi^\varepsilon)$ satisfying

$$\begin{aligned} &\|\rho^\varepsilon(t) - 1\|_m^2 + \|u^\varepsilon(t)\|_m^2 + \varepsilon\,\|\pi^\varepsilon(t)\|_m^2 \\ &+ \int_0^t \big(\|\nabla\rho^\varepsilon(t')\|_{m-1}^2 + \|\nabla u^\varepsilon(t')\|_{m-1}^2 + \|\pi^\varepsilon(t')\|_m^2\big)dt' \\ \le\ & c\big(\|\rho_0^\varepsilon - 1\|_m^2 + \|u_0^\varepsilon\|_m^2 + \varepsilon\|\pi_0^\varepsilon\|_m^2\big), \quad \forall\, t \ge 0. \end{aligned} \tag{6.7}$$

Moreover, there exist functions (ρ, u, π) with $(\rho - 1, u) \in L^\infty(\mathbb{R}^+; H^m)$ and $\pi \in L^2(\mathbb{R}^+; H^m)$, such that, as $\varepsilon \to 0$ and up to subsequences,

$$\begin{aligned} (\rho^\varepsilon, u^\varepsilon) &\longrightarrow (\rho, u), \quad \textit{weakly-}* \textit{ in } L^\infty(\mathbb{R}^+; H^m), \\ \pi^\varepsilon &\longrightarrow \pi, \quad \textit{weakly in } L^2(\mathbb{R}^+; H^m), \end{aligned}$$

where

$$\pi = -\nu\sigma(u) - \lambda(\operatorname{div} u) I_d,$$

and (ρ, u) is a unique solution to (2.11) for the compressible Navier-Stokes equations with initial value (ρ_0, u_0) being the weak limit of $(\rho_0^\varepsilon, u_0^\varepsilon)$ in H^m (up to subsequences).

Proof. Let $T > 0$. We follow the same steps in the proof of Theorem 4.1 by considering energy estimates for V_1 defined on $[0, T]$. Integrating (6.5) over $\mathbb{R}^d$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} E_1(V_1)dx + 2\int_{\mathbb{R}^d} P_1(\pi)dx = 0, \tag{6.8}$$

which provides an L^2 estimate for V_1 with dissipation for π.

Next, for $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le m$, applying ∂_x^α to (6.1) and taking the inner product in L^2 with

$$\big(2h'(\rho)\partial_x^\alpha\rho, 2\rho\partial_x^\alpha u, \nu^{-1}\partial_x^\alpha\pi\big),$$

we obtain an energy equality for V_1. Then combining this equality with the energy for $\operatorname{tr}(\partial_x^\alpha\pi)$, similarly to (4.10) and (6.5), we have

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d}\Big(h'(\rho)|\partial_x^\alpha\rho|^2 + \rho|\partial_x^\alpha u|^2 + 2\varepsilon P_1(\partial_x^\alpha\pi)\Big)dx + 4\int_{\mathbb{R}^d} P_1(\partial_x^\alpha\pi)dx \\ =\ & \langle \operatorname{div}\vec{B}(U_1)\partial_x^\alpha U_1, \partial_x^\alpha U_1\rangle - 2\sum_{j=1}^d K_j^\alpha - 2\langle\partial_x^\alpha(\rho^{-1}\operatorname{div}\pi) - \rho^{-1}\partial_x^\alpha\operatorname{div}\pi, \rho\partial_x^\alpha u\rangle, \end{aligned}$$

where the first two terms on the right-hand side are estimated in (4.14) and (4.16)-(4.17) with $U_1 = (\rho, u)$. For the last term, since $|\alpha| \ge 1$, the Moser-type calculus inequalities yield

$$\big|\langle\partial_x^\alpha(\rho^{-1}\operatorname{div}\pi - \rho^{-1}\operatorname{div}\partial_x^\alpha\pi, \rho\partial_x^\alpha u\rangle\big| \le c\|\nabla\rho\|_{m-1}\|\nabla u\|_{m-1}\|\nabla\pi\|_{m-1}.$$

It follows that

$$\begin{aligned}(6.9)\qquad &\frac{d}{dt}\int_{\mathbb{R}^d}\Big(h'(\rho)|\partial_x^\alpha\rho|^2+\rho|\partial_x^\alpha u|^2+2\varepsilon P_1(\partial_x^\alpha\pi)\Big)dx+4\int_{\mathbb{R}^d}P_1(\partial_x^\alpha\pi)dx\\ \le\;& c\big(\|\nabla\rho\|_{m-1}+\|\nabla u\|_{m-1}\big)\|\nabla V_1\|_{m-1}^2.\end{aligned}$$

Adding (6.9) for all α with $1\le|\alpha|\le m$, together with (6.8), we obtain

$$\mathcal{E}_1'(t)+4\mathcal{D}_1(t)\le c\big(\|\nabla\rho\|_{m-1}+\|\nabla u\|_{m-1}\big)\|\nabla V_1\|_{m-1}^2. \tag{6.10}$$

where $\mathcal{E}_1(t)$ is the total energy defined by

$$\mathcal{E}_1(t)=2\int_{\mathbb{R}^d}E_0(\rho,u)dx+\sum_{1\le|\alpha\le m}\int_{\mathbb{R}^d}\Big(h'(\rho)|\partial_x^\alpha\rho|^2+\rho|\partial_x^\alpha u|^2\Big)dx+2\varepsilon\mathcal{D}_1(t),$$

with

$$\mathcal{D}_1(t)=\sum_{|\alpha|\le m}\int_{\mathbb{R}^d}P_1(\partial_x^\alpha\pi)dx.$$

From (6.3)-(6.4) and (6.6), it is easy to see that $\mathcal{D}_1(t)$ is uniformly equivalent to $\|\pi\|_m^2$, and $\mathcal{E}_1(t)$ is uniformly equivalent to $\|\rho-1\|_m^2+\|u\|_m^2+\varepsilon\|\pi\|_m^2$. This provides an estimate in H^m for V_1 with dissipation for π.

Finally, we consider a dissipation estimate for $\nabla\rho$ and ∇u. Let $\beta\in\mathbb{N}^d$ with $|\beta|\le m-1$. Applying $\operatorname{div}\partial_x^\beta$ to the third equation in (6.1), we have

$$\varepsilon\partial_t(\partial_x^\beta\operatorname{div}\pi)+\nu\Delta\partial_x^\beta u+\mu\nabla(\operatorname{div}\partial_x^\beta u)=-\partial_x^\beta\operatorname{div}\pi,$$

where $\mu=\nu+\lambda-\frac{2\nu}{d}$. Taking the inner product with $\partial_x^\beta u$ in L^2, we have

$$\nu\|\nabla(\partial_x^\beta u)\|^2+\mu\|\operatorname{div}(\partial_x^\beta u)\|^2=\varepsilon\langle\partial_t(\partial_x^\beta\operatorname{div}\pi),\partial_x^\beta u\rangle-\langle\partial_x^\beta\pi,\nabla\partial_x^\beta u\rangle,$$

which is a key relation to obtain the dissipation for $\nabla\rho$ and ∇u. Let $2\kappa=\min(\nu,\lambda)>0$. We know that

$$\nu\|\nabla(\partial_x^\beta u)\|^2+\mu\|\operatorname{div}(\partial_x^\beta u)\|^2\ge2\kappa\|\nabla(\partial_x^\beta u)\|^2.$$

Clearly,

$$\big|\langle\nabla\partial_x^\beta u,\partial_x^\beta\pi\rangle\big|\le\kappa\|\nabla(\partial_x^\beta u)\|^2+c\|\partial_x^\beta\pi\|^2,$$

Therefore,

$$\kappa\|\partial_x^\beta(\nabla u)\|^2\le c\|\pi\|_m^2+\varepsilon\langle\partial_t(\partial_x^\beta\operatorname{div}\pi),\partial_x^\beta u\rangle.$$

The rest of the proof is similar to that of (4.18). By using analogous techniques, we obtain

$$\langle\partial_t(\partial_x^\beta\operatorname{div}\pi),\partial_x^\beta u\rangle=\frac{d}{dt}\langle\partial_x^\beta\operatorname{div}\pi,\partial_x^\beta u\rangle+\langle\partial_x^\beta((u\cdot\nabla)u+\nabla h(\rho)+\rho^{-1}\operatorname{div}\pi),\partial_x^\beta\operatorname{div}\pi\rangle,$$

$$\begin{aligned}\|\partial_x^\beta\nabla h(\rho)\|^2 \;=\;& -\frac{d}{dt}\langle\partial_x^\beta\nabla h(\rho),\partial_x^\beta u\rangle+\langle\partial_x^\beta(h'(\rho)\operatorname{div}(\rho u)),\partial_x^\beta\operatorname{div}u\rangle\\ &-\langle\partial_x^\beta((u\cdot\nabla)u+\rho^{-1}\operatorname{div}\pi),\partial_x^\beta\nabla h(\rho)\rangle.\end{aligned}$$

Combining the last three relations yields

$$\begin{aligned}&\frac{d}{dt}\langle(2\eta_1\partial_x^\beta\nabla h(\rho)-\varepsilon\partial_x^\beta\operatorname{div}\pi),\partial_x^\beta u\rangle+\kappa\|\partial_x^\beta(\nabla u)\|^2+\eta_1\|\partial_x^\beta\nabla h(\rho)\|^2\\ \le\;& c\|\pi\|_m^2+c\big(\|\nabla\rho\|_{m-1}+\|u\|_m\big)\|\nabla V_1\|_{m-1}^2,\end{aligned}$$

where $\eta_1 > 0$ is a sufficiently small constant. Adding this inequality for all $|\beta| \le m-1$, together with (6.10), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\mathcal{E}_1(t) + \eta_2 \sum_{|\beta|\le m-1} \langle (2\eta_1 \partial_x^\beta \nabla h(\rho) - \varepsilon \partial_x^\beta \operatorname{div} \pi), \partial_x^\beta u\rangle\right) \\
&+ 4\mathcal{D}_1(t) + \eta_2\Big(\kappa \|\nabla u\|_{m-1}^2 + \eta_1 \|\nabla h(\rho)\|_{m-1}^2\Big) \\
\le\ & c\|\pi\|_m^2 + c\big(\|\nabla\rho\|_{m-1} + \|u\|_m\big)\|\nabla V_1\|_{m-1}^2,
\end{aligned}$$

where $\eta_2 > 0$ is a small constant to be chosen. Integration this inequality over $[0,t]$ with $t > 0$ and taking $\eta_2 > 0$ sufficiently small, as in the proof of Lemma 4.5, we obtain

$$\begin{aligned}
&\|\rho(t)-1\|_m^2 + \|u(t)\|_m^2 + \varepsilon\|\pi(t)\|_m^2 + \int_0^t \big(\|\nabla\rho(t')\|_{m-1}^2 + \|\nabla u(t')\|_{m-1}^2 + \|\pi(t')\|_m^2\big)dt' \\
\le\ & c\big(\|\rho_0 - 1\|_m^2 + \|u_0\|_m^2 + \varepsilon\|\pi_0\|_m^2\big), \quad \forall\, t \in [0,T],
\end{aligned}$$

which implies (6.7) and the uniform global existence of solutions. The global convergence of system (6.1) can be performed in a similar way to the proof of Theorem 4.1. □

6.2. **The system with revised Maxwell's constitutive relations.**

We consider system (1.7) with the revised Maxwell's constitutive relations, namely,

$$\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) + \operatorname{div}\pi_1 + \nabla\pi_2 = 0, \\
\varepsilon_1 \partial_t \pi_1 + \nu\sigma(u) = -\pi_1, \\
\varepsilon_2 \partial_t \pi_2 + \lambda \operatorname{div} u = -\pi_2, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d.
\end{cases} \tag{6.11}$$

with initial condition

$$t = 0: \quad (\rho, u, \pi_1, \pi_2) = (\rho_0^\varepsilon, u_0^\varepsilon, \pi_{10}^\varepsilon, \pi_{20}^\varepsilon), \tag{6.12}$$

where π_1 is a square matrix variable of order d, π_2 is a scalar variable and $\varepsilon = (\varepsilon_1, \varepsilon_2)$. Similarly to π in (6.1), we may assume that π_1 is symmetric and we know that this assumption is not a restriction condition.

As mentioned in the introduction, $\operatorname{tr}(\pi_1)$ satisfies a linear equation

$$\varepsilon_1 \partial_t \operatorname{tr}(\pi_1) + \operatorname{tr}(\pi_1) = 0, \tag{6.13}$$

and $\operatorname{tr}(\pi_1(t,\cdot)) = 0$ for all $t > 0$ if and only if $\operatorname{tr}(\pi_{10}^\varepsilon) = 0$. Condition $\operatorname{tr}(\pi_{10}^\varepsilon) = 0$ is used in the study of (6.11)-(6.12) in [39]. However, it is a real restriction on the initial data. In this subsection, we consider the Cauchy problem (6.11)-(6.12) without this condition. The local convergence of system (6.11) can be proved in a similar way to the proof of Theorem 3.1 as $\varepsilon_1 = \varepsilon_2 \to 0$. Now we establish the global convergence of the system near a constant state $(1,0,0,0)$ for $V_2 = (\rho, u, \pi_1, \pi_2)$.

We start with the definition of a pair of entropy-entropy flux of the system. From (6.1) and (6.11), we have successively

$$\begin{aligned}
&\partial_t E_0(\rho,u) + \operatorname{div} F_0(\rho,u) + u\cdot(\operatorname{div}\pi_1 + \nabla\pi_2) = 0, \\
&\frac{\varepsilon_1}{4\nu}\partial_t|\pi_1|^2 + \frac{1}{2\nu}|\pi_1|^2 + \frac{1}{2}\pi_1 : \sigma(u) = 0, \\
&\frac{\varepsilon_2}{2\lambda}\partial_t|\pi_2|^2 + \frac{1}{\lambda}|\pi_2|^2 + \pi_2 \operatorname{div} u = 0.
\end{aligned}$$

Since

$$\frac{1}{2}\pi_1 : \sigma(u) = \pi_1 : \nabla u - \frac{1}{d}(\operatorname{div} u)\operatorname{tr}(\pi_1),$$

we obtain

$$\partial_t\Big(E_0(\rho,u) + \frac{\varepsilon_1}{4\nu}|\pi_1|^2 + \frac{\varepsilon_2}{2\lambda}|\pi_2|^2\Big) + \operatorname{div}\big(F_0(\rho,u) + (\pi_1 + \pi_2 I_d)u\big)$$
$$+ \frac{1}{2\nu}|\pi_1|^2 + \frac{1}{\lambda}|\pi_2|^2 - \frac{1}{d}(\operatorname{div} u)\operatorname{tr}(\pi_1) = 0.$$

In order to eliminate the last term on the right hand-side of the above equality, we multiply the last equation in (6.11) by $\operatorname{tr}(\pi_1)/d\lambda$ and (6.13) by $\varepsilon_2\pi_2/\varepsilon_1 d\lambda$ to yield

$$\frac{\varepsilon_2}{d\lambda}\operatorname{tr}(\pi_1)\partial_t\pi_2 + \frac{1}{d}(\operatorname{div} u)\operatorname{tr}(\pi_1) + \frac{1}{d\lambda}\pi_2\operatorname{tr}(\pi_1) = 0,$$
$$\frac{\varepsilon_2}{d\lambda}\pi_2\partial_t\operatorname{tr}(\pi_1) + \frac{\varepsilon_2}{d\lambda\varepsilon_1}\pi_2\operatorname{tr}(\pi_1) = 0.$$

This implies that

$$\frac{\varepsilon_2}{d\lambda}\partial_t\big[\pi_2\operatorname{tr}(\pi_1)\big] + \frac{1}{d}(\operatorname{div} u)\operatorname{tr}(\pi_1) + \frac{1}{d\lambda}\Big(\frac{\varepsilon_2}{\varepsilon_1} + 1\Big)\pi_2\operatorname{tr}(\pi_1) = 0.$$

Thus,

$$\partial_t\Big(E_0(\rho,u) + \frac{\varepsilon_1}{4\nu}|\pi_1|^2 + \frac{\varepsilon_2}{2\lambda}|\pi_2|^2 + \frac{\varepsilon_2}{d\lambda}\pi_2\operatorname{tr}(\pi_1)\Big) + \operatorname{div}\big(F_0(\rho,u) + (\pi_1 + \pi_2 I_d)u\big)$$
$$+ \frac{1}{2\nu}|\pi_1|^2 + \frac{1}{\lambda}|\pi_2|^2 + \frac{1}{d\lambda}\Big(\frac{\varepsilon_2}{\varepsilon_1} + 1\Big)\pi_2\operatorname{tr}(\pi_1) = 0. \tag{6.14}$$

This provides a pair of entropy-entropy flux. However, the strict convexity of the entropy and the negativity of the entropy production are not guaranteed. To remedy this, we use again (6.13) to obtain

$$2a(\varepsilon)\varepsilon_1\partial_t|\operatorname{tr}(\pi_1)|^2 + 4a(\varepsilon)|\operatorname{tr}(\pi_1)|^2 = 0, \tag{6.15}$$

where $a(\varepsilon) > 0$ is a constant defined by

$$a(\varepsilon) = \frac{1}{4d^2\lambda}\Big(\frac{\varepsilon_2}{\varepsilon_1} + 1\Big)^2.$$

Finally, adding (6.14) and (6.15), it yields

$$\partial_t E_2(V_2) + \operatorname{div} F_2(V_2) + P_2(\pi_1,\pi_2) = 0, \tag{6.16}$$

where $V_2 = (\rho, u, \pi_1, \pi_2)$,

$$\begin{cases} E_2(V_2) = E_0(\rho,u) + \tilde{E}_2(\pi_1,\pi_2), \\ F_2(V_2) = F_0(\rho,u) + (\pi_1 + \pi_2 I_d)u, \\ P_2(\pi_1,\pi_2) = \dfrac{1}{2\nu}|\pi_1|^2 + \dfrac{1}{\lambda}|\pi_2|^2 + 4a(\varepsilon)|\operatorname{tr}(\pi_1)|^2 + \dfrac{1}{d\lambda}\Big(\dfrac{\varepsilon_2}{\varepsilon_1} + 1\Big)\pi_2\operatorname{tr}(\pi_1), \end{cases} \tag{6.17}$$

with

$$\tilde{E}_2(\pi_1,\pi_2) = \frac{\varepsilon_1}{4\nu}|\pi_1|^2 + \frac{\varepsilon_2}{2\lambda}|\pi_2|^2 + 2a(\varepsilon)\varepsilon_1|\operatorname{tr}(\pi_1)|^2 + \frac{\varepsilon_2}{d\lambda}\pi_2\operatorname{tr}(\pi_1).$$

By the Young inequality, we see that

$$\frac{\varepsilon_1}{4\nu}|\pi_1|^2 + \frac{\varepsilon_2}{4\lambda}|\pi_2|^2 \le \tilde{E}_2(\pi_1,\pi_2) \le \frac{\varepsilon_1}{4\nu}|\pi_1|^2 + \frac{3\varepsilon_2}{4\lambda}|\pi_2|^2 + \frac{2}{d^2\lambda}\Big(\frac{\varepsilon_2^2}{\varepsilon_1} + \varepsilon_1\Big)|\operatorname{tr}(\pi_1)|^2 \tag{6.18}$$

and

$$P_2(\pi_1,\pi_2) \ge \frac{1}{2\nu}|\pi_1|^2 + \frac{1}{2\lambda}|\pi_2|^2. \tag{6.19}$$

Therefore, E_2 is a strictly convex entropy with respect to $(\rho, \rho u, \pi_1, \pi_2)$ for $\rho > 0$ and the entropy production $-P_2$ is negative.

The main result of this subsection is stated as follows.

Theorem 6.2. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2)$ and $m > \frac{d}{2}+1$ be an integer. Assume $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \pi_{10}^\varepsilon, \pi_{20}^\varepsilon) \in H^m$ with π_{10}^ε being symmetric. There are two positive constants δ and c (independent of ε) such that if*

$$\|\rho_0^\varepsilon - 1\|_m + \|u_0^\varepsilon\|_m + \sqrt{\varepsilon_1}\,\|\pi_{10}^\varepsilon\|_m + \sqrt{\varepsilon_2}\,\|\pi_{20}^\varepsilon\|_m + \frac{\varepsilon_2}{\sqrt{\varepsilon_1}}\|tr(\pi_{10}^\varepsilon)\|_m \le \delta,$$

then for all $\varepsilon_1, \varepsilon_2 \in (0, 1]$, the Cauchy problem (6.11)-(6.12) admits a unique global solution $(\rho^\varepsilon, u^\varepsilon, \pi_1^\varepsilon, \pi_2^\varepsilon)$ satisfying

$$\begin{aligned} &\|\rho^\varepsilon(t) - 1\|_m^2 + \|u^\varepsilon(t)\|_m^2 + \varepsilon_1\,\|\pi_1^\varepsilon(t)\|_m^2 + \varepsilon_2\,\|\pi_2^\varepsilon(t)\|_m^2 + \frac{\varepsilon_2^2}{\varepsilon_1}\|tr(\pi_1^\varepsilon(t))\|_m^2 \\ &+ \int_0^t \big(\|\nabla\rho^\varepsilon(t')\|_{m-1}^2 + \|\nabla u^\varepsilon(t')\|_{m-1}^2 + \|\pi_1^\varepsilon(t')\|_m^2 + \|\pi_2^\varepsilon(t')\|_m^2\big)dt' \\ &\le\ c\Big(\|\rho_0^\varepsilon - 1\|_m^2 + \|u_0^\varepsilon\|_m^2 + \varepsilon_1\|\pi_{10}^\varepsilon\|_m^2 + +\varepsilon_2\|\pi_{20}^\varepsilon\|_m^2 + \frac{\varepsilon_2^2}{\varepsilon_1}\|tr(\pi_{10}^\varepsilon)\|_m^2\Big), \quad \forall\, t \ge 0. \end{aligned} \tag{6.20}$$

Moreover, there exist functions $(\rho - 1, u) \in L^\infty(\mathbb{R}^+; H^m)$ and $\pi_1, \pi_2 \in L^2(\mathbb{R}^+; H^m)$, such that, as $(\varepsilon_1, \varepsilon_2) \to 0$ and up to subsequences,

$$\begin{aligned} (\rho^\varepsilon, u^\varepsilon) &\longrightarrow (\rho, u), \quad \textit{weakly-}* \ \textit{in } L^\infty(\mathbb{R}^+; H^m), \\ (\pi_1^\varepsilon, \pi_2^\varepsilon) &\longrightarrow (\pi_1, \pi_2), \quad \textit{weakly in } L^2(\mathbb{R}^+; H^m), \end{aligned}$$

where

$$\pi_1 = -\nu\sigma(u), \quad \pi_2 = -\lambda \operatorname{div} u,$$

and (ρ, u) is a unique solution to (2.11) for the compressible Navier-Stokes equations with initial value (ρ_0, u_0) being the weak limit of $(\rho_0^\varepsilon, u_0^\varepsilon)$ in H^m (up to subsequences).

Proof. The proof is similar to that of Theorem 6.1. It follows from energy and dissipation estimates that we give below. The detail of the proof is omitted here. Let $T > 0$ and V_2 be a smooth solution defined on $[0, T]$.

Firstly, the entropy equality (6.16) yields

$$\frac{d}{dt}\int_{\mathbb{R}^d} E_2(V_2)dx + \int_{\mathbb{R}^d} P_2(\pi_1, \pi_2)dx = 0, \tag{6.21}$$

which provides an L^2 estimate for V_2 with dissipation for π_1 and π_2. Next, let $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le m$. Applying ∂_x^α to (6.11) and taking the inner product in L^2 with

$$\big(2h'(\rho)\partial_x^\alpha\rho, 2\rho\partial_x^\alpha u, \nu^{-1}\partial_x^\alpha\pi_1, 2\lambda^{-1}\partial_x^\alpha\pi_2\big),$$

together with the energy equality for $\mathrm{tr}(\partial_x^\alpha\pi_1)$, we obtain an energy equality for V_2 :

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d}\Big(h'(\rho)|\partial_x^\alpha\rho|^2 + \rho|\partial_x^\alpha u|^2 + 2\tilde{E}_2(\partial_x^\alpha\pi_1, \partial_x^\alpha\pi_2)\Big)dx + 2\int_{\mathbb{R}^d} P_2(\partial_x^\alpha\pi_1, \partial_x^\alpha\pi_2)dx \\ =\ &\langle \operatorname{div}\vec{B}(U_1)\partial_x^\alpha U_1, \partial_x^\alpha U_1\rangle - 2\sum_{j=1}^d K_j^\alpha - 2\langle\partial_x^\alpha(\rho^{-1}\operatorname{div}\pi_1) - \rho^{-1}\partial_x^\alpha \operatorname{div}\pi_1, \rho\partial_x^\alpha u\rangle \\ &- 2\langle\partial_x^\alpha(\rho^{-1}\nabla\pi_2) - \rho^{-1}\partial_x^\alpha\nabla\pi_2, \rho\partial_x^\alpha u\rangle, \end{aligned}$$

where the first two terms on the right-hand side are estimated in (4.14) and (4.16)-(4.17). Together with (6.21), we further obtain

$$\mathcal{E}_2'(t) + 2\mathcal{D}_2(t) \le c\big(\|\nabla\rho\|_{m-1} + \|\nabla u\|_{m-1}\big)\|\nabla V_2\|_{m-1}^2, \tag{6.22}$$

where $\mathcal{E}_2(t)$ is the total energy defined by

$$\mathcal{E}_2(t) = 2\int_{\mathbb{R}^d} E_0(\rho,u)dx + \sum_{1\le|\alpha|\le m}\int_{\mathbb{R}^d}\Big(h'(\rho)|\partial_x^\alpha\rho|^2 + \rho|\partial_x^\alpha u|^2\Big)dx + 2\sum_{|\alpha|\le m}\int_{\mathbb{R}^d}\tilde{E}_2(\partial_x^\alpha\pi_1,\partial_x^\alpha\pi_2)dx,$$

and

$$\mathcal{D}_2(t) = \sum_{|\alpha|\le m}\int_{\mathbb{R}^d} P_2(\partial_x^\alpha\pi_1,\partial_x^\alpha\pi_2)dx.$$

From (6.19), it is easy to see that

$$\mathcal{D}_2(t) \ge \frac{1}{2\nu}\|\pi_1|_m^2 + \frac{1}{2\lambda}\|\pi_2|_m^2.$$

Thus, (6.22) together with (6.17)-(6.18) provides an estimate in H^m for V_2 with dissipation for π_1 and π_2.

For the dissipation estimate of $\nabla\rho$ and ∇u, let $\beta\in\mathbb{N}^d$ with $|\beta|\le m-1$. From the equations in (6.11), we obtain successively

$$\begin{aligned}\|\partial_x^\beta\nabla h(\rho)\|^2 &= -\frac{d}{dt}\langle\partial_x^\beta\nabla h(\rho),\partial_x^\beta u\rangle + \langle\partial_x^\beta(h'(\rho)\operatorname{div}(\rho u)),\partial_x^\beta\operatorname{div} u\rangle\\ &\quad - \langle\partial_x^\beta((u\cdot\nabla)u+\rho^{-1}(\operatorname{div}\pi_1+\nabla\pi_2)),\partial_x^\beta\nabla h(\rho)\rangle,\end{aligned}$$

$$\begin{aligned}\nu\|\nabla(\partial_x^\beta u)\|^2 + \nu\Big(1-\frac{2}{d}\Big)\|\operatorname{div}(\partial_x^\beta u)\|^2 &= \varepsilon_1\langle\partial_t(\partial_x^\beta\operatorname{div}\pi_1),\partial_x^\beta u\rangle - \langle\partial_x^\beta\pi_1,\nabla\partial_x^\beta u\rangle,\\ \lambda\|\operatorname{div}(\partial_x^\beta u)\|^2 &= \varepsilon_2\langle\partial_t(\partial_x^\beta\nabla\pi_2),\partial_x^\beta u\rangle - \langle\partial_x^\beta\pi_2,\operatorname{div}\partial_x^\beta u\rangle,\end{aligned}$$

$$\begin{aligned}\langle\partial_t\partial_x^\beta(\varepsilon_1\operatorname{div}\pi_1+\varepsilon_2\nabla\pi_2),\partial_x^\beta u\rangle &= \frac{d}{dt}\langle\partial_x^\beta(\varepsilon_1\operatorname{div}\pi_1+\varepsilon_2\nabla\pi_2),\partial_x^\beta u\rangle\\ &\quad + \langle\partial_x^\beta(\varepsilon_1\operatorname{div}\pi_1+\varepsilon_2\nabla\pi_2),\partial_x^\beta((u\cdot\nabla)u+\rho^{-1}(\operatorname{div}\pi_1+\nabla\pi_2))\rangle\\ &\quad + \langle\partial_x^\beta(\varepsilon_1\operatorname{div}\pi_1+\varepsilon_2\nabla\pi_2),\partial_x^\beta(\nabla h(\rho))\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{d}{dt}\left(\mathcal{E}_2(t)+\eta_2\sum_{|\beta|\le m-1}\langle\partial_x^\beta(2\eta_1\nabla h(\rho)-\varepsilon_1\operatorname{div}\pi_1-\varepsilon_2\nabla\pi_2),\partial_x^\beta u\rangle\right)\\ &\quad + 2\mathcal{D}_2(t) + \eta_2\Big(\kappa\|\nabla u\|_{m-1}^2 + \eta_1\|\nabla h(\rho)\|_{m-1}^2\Big)\\ &\le\ c\big(\|\pi_1\|_m^2+\|\pi_2\|_m^2\big) + c\big(\|\nabla\rho\|_{m-1}+\|u\|_m\big)\|\nabla V_2\|_{m-1}^2.\end{aligned}$$

This estimate implies (6.20) and then the result in Theorem 6.2 follows. □

6.3. **Incompressible case.**

Let $d\ge 2$. In the incompressible case, both systems (6.1) and (6.11) lead to the following system

$$\begin{cases}\partial_t u + (u\cdot\nabla)u + \nabla p + \operatorname{div}\pi = 0,\\ \varepsilon\partial_t\pi + \nu\big(\nabla u + (\nabla u)^T\big) = -\pi, \quad \text{in } \mathbb{R}^+\times\mathbb{R}^d\end{cases} \tag{6.23}$$

and

$$\operatorname{div} u = 0, \tag{6.24}$$

with initial condition

$$t=0: \quad (u,\pi) = (u_0^\varepsilon,\pi_0^\varepsilon), \tag{6.25}$$

where π is a square matrix variable of order d and $\varepsilon > 0$ is a small parameter. Obviously, (6.23) is an approximation of the incompressible Navier-Stokes equations (5.1)-(5.2). Applying div div to the second equation in (6.23), we have

$$\varepsilon\partial_t(\operatorname{div}\operatorname{div}\pi) + \operatorname{div}\operatorname{div}\pi = 0.$$

Suppose now

$$\operatorname{div} u_0^\varepsilon = 0, \quad \operatorname{div}\operatorname{div}\pi_0^\varepsilon = 0. \tag{6.26}$$

We obtain $\operatorname{div}\operatorname{div}\pi(t,\cdot) = 0$ for all time $t > 0$. In this case, we still have

$$-\Delta p = \operatorname{tr}\big((\nabla u)^2\big). \tag{6.27}$$

Similarly to Proposition 5.1, we have the following result of which the proof is omitted.

Proposition 6.1. *Let $T > 0$ and $(u_0^\varepsilon, \pi_0^\varepsilon) \in H^2$ satisfying (6.26). Let (u, p, π) be a solution of (6.23)-(6.25) with regularity*

$$(u,\pi) \in L^\infty(0,T;H^2) \cap W^{1,\infty}(0,T;H^1), \quad \nabla u \in L^\infty\big((0,T)\times\mathbb{R}^d\big), \quad p \in L^\infty(0,T;H^2).$$

Then the incompressibility condition (6.24) is equivalent to (6.27) in $(0,T)\times\mathbb{R}^d$. As a consequence, we also have $\operatorname{div}\operatorname{div}\pi = 0$ in $(0,T)\times\mathbb{R}^d$.

Similarly to Theorems 5.1-5.2, we can prove the local and global convergences of system (6.23)-(6.24) to incompressible Navier-Stokes equations (5.1)-(5.2). The detail is omitted here.

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