



HAL
open science

LIE-RINEHART AND HOCHSCHILD COHOMOLOGY FOR ALGEBRAS OF DIFFERENTIAL OPERATORS

Thierry Lambre, Francisco Kordon

► **To cite this version:**

Thierry Lambre, Francisco Kordon. LIE-RINEHART AND HOCHSCHILD COHOMOLOGY FOR ALGEBRAS OF DIFFERENTIAL OPERATORS. Journal of Pure and Applied Algebra, Elsevier, 2020, 225, 10.1016/j.jpaa.2020.106456 . hal-03446945

HAL Id: hal-03446945

<https://hal.uca.fr/hal-03446945>

Submitted on 24 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives | 4.0
International License

LIE–RINEHART AND HOCHSCHILD COHOMOLOGY FOR ALGEBRAS OF DIFFERENTIAL OPERATORS

FRANCISCO KORDON AND THIERRY LAMBRE

ABSTRACT. Let (S, L) be a Lie–Rinehart algebra such that L is S -projective and let U be its universal enveloping algebra. In this paper we present a spectral sequence which converges to the Hochschild cohomology of U with values on a U -bimodule M and whose second page involves the Lie–Rinehart cohomology of the algebra and the Hochschild cohomology of S with values on M . After giving a convenient description of the involved algebraic structures we use the spectral sequence to compute explicitly the Hochschild cohomology of the algebra of differential operators tangent to a central arrangement of three lines.

INTRODUCTION

The goal of this article is to apply homological algebra techniques for Lie–Rinehart algebras to a problem of algebras of differential operators. We begin by describing a spectral sequence that converges to the Hochschild cohomology of the enveloping algebra of a Lie–Rinehart algebra. After that, we focus on the algebra of differential operators $\text{Diff}\mathcal{A}$ associated to a central arrangement \mathcal{A} of three lines. This is a graded associative algebra that is at the same time the enveloping algebra of a Lie–Rinehart algebra: an explicit calculation with the spectral sequence allows us to compute the Hilbert series of its Hochschild cohomology. We conclude by giving two other examples of algebras in which the spectral sequence proves useful.

Let \mathbb{k} be a field of characteristic zero and let \mathcal{A} be a central hyperplane arrangement in a finite dimensional \mathbb{k} -vector space V . Let S be the algebra of coordinates on V and let $Q \in S$ be a defining polynomial for \mathcal{A} . The arrangement \mathcal{A} is free if the Lie algebra $\text{Der}\mathcal{A} = \{\theta \in \text{Der}S : \theta(Q) \in QS\}$ of derivations of S tangent to \mathcal{A} is a free S -module. It is not known what makes an arrangement free, but this condition is nevertheless satisfied in many important examples; for instance, it is a theorem by H. Terao in [18] that reflection arrangements over \mathbb{C} are free. We refer to P. Orlik and H. Terao’s book [15] for a general reference of hyperplane arrangements.

The algebra $\text{Diff}\mathcal{A}$ of differential operators tangent to an arrangement \mathcal{A} , first considered by F. J. Calderón-Moreno in [5], is the algebra of differential operators on S which preserve the ideal QS of S and all its powers. We are interested in the Hochschild cohomology of $\text{Diff}\mathcal{A}$ when \mathcal{A} is free.

Date: June 3, 2020.

The first and simplest example of a free arrangement is that of a central line arrangement, that is, when $V = \mathbb{k}^2$. Let l be the number of lines of such an arrangement: for $l \geq 5$, the Hochschild cohomology of $\text{Diff}\mathcal{A}$ has been obtained as a Gerstenhaber algebra by the first author and M. Suárez-Álvarez in [9] starting from a projective resolution of $\text{Diff}\mathcal{A}$ as a bimodule over itself by means of explicit calculations that exploit a graded algebra structure on $\text{Diff}\mathcal{A}$, but the calculations performed in this situation seem impossible to emulate when $l = 3$ or $l = 4$. In this paper we are able to extend, in Corollaries 5.9 and 5.10, some of these results to the most complicated case, which is when $l = 3$:

Theorem A. *Let \mathcal{A} be a central arrangement of three lines. The Hilbert series of $HH^\bullet(\text{Diff}\mathcal{A})$ is $h_{HH^\bullet(\text{Diff}\mathcal{A})}(t) = 1 + 3t + 6t^2 + 4t^3$. The first cohomology space $HH^1(\text{Diff}\mathcal{A})$ is an abelian Lie algebra of dimension three.*

It is to prove Theorem A that Lie–Rinehart algebras come to into play: the pair $(S, \text{Der}\mathcal{A})$ is a Lie–Rinehart algebra. Recall that a Lie–Rinehart algebra (S, L) consists of a commutative algebra S and a Lie algebra L with an S -module structure that acts on S by derivations and which satisfies certain compatibility conditions analogous to those satisfied by the pair $(S, \text{Der}S)$. The universal enveloping algebra U of a Lie–Rinehart algebra (S, L) and the Lie–Rinehart cohomology $H_S^\bullet(L, N) = \text{Ext}_U^\bullet(S, N)$ are an associative algebra and a cohomology theory that generalize the usual enveloping algebra and the Lie algebra cohomology of the Lie algebra L by taking into account its interaction with S —see the original paper [16] by G. Rinehart or the more modern exposition [8] by J. Huebschmann.

If \mathcal{A} is free, as remarked by L. Narváez Macarro in [12, Theorem 1.3.1], the enveloping algebra of $(S, \text{Der}\mathcal{A})$ is isomorphic to $\text{Diff}\mathcal{A}$. To compute the Hochschild cohomology in Theorem A above we employ a strategy that gives rise to a general method to approach this kind of computations: we construct, in Corollary 3.3, a spectral sequence converging to the Hochschild cohomology $H^\bullet(U, M)$ of the enveloping algebra U with values on an U -bimodule M . For this sequence we need an U -module structure on $H^\bullet(S, M)$, the Hochschild cohomology of S with values on M . This U -module structure is constructed using an injective resolution of M by U -bimodules and we see in Theorem 2.8 that it can be computed explicitly from a projective resolution of S by S -bimodules. Moreover, the action of each $\alpha \in L$ on $H^\bullet(S, M)$, computed using projectives, by the endomorphism ∇_α^\bullet given in Remark 2.5 turns out suitable for computations.

Theorem B. *Let (S, L) be a Lie–Rinehart pair such that L is an S -projective module and let M be an U -bimodule. There exist a U -module structure on $H^\bullet(S, M)$ and a first-quadrant spectral sequence E_\bullet converging to $H^\bullet(U, M)$ with second page*

$$E_2^{p,q} = H_S^p(L, H^q(S, M)).$$

We give two other applications of Theorem B. First, in Subsection 6.1 we compute the Hochschild cohomology of a family of subalgebras of the Weyl algebra over a field of characteristic zero, that is, the algebras A_h generated by elements x and y satisfying

the relation $yx - xy = h$ for a given $h \in \mathbb{k}[x]$. These algebras have been studied by G. Benkart, S. Lopes and M. Ondrus in the series of articles that start with [2] for a field of arbitrary characteristic and, more recently, S. Lopes and A. Solotar in [11] have described their Hochschild cohomology, with special emphasis on the Lie module structure of the second cohomology space over the first one, also in arbitrary characteristic. Some of the expressions we provide were nevertheless not found before and might be of interest. Second, in Subsection 6.2 we recover in a more direct and clear way a result by the second author and P. Le Meur in [10] that states that the enveloping algebra U of a Lie-Rinehart algebra (S, L) has Van den Bergh duality in dimension $n + d$ if S has Van den Bergh duality in dimension n and L is finitely generated and projective with constant rank d .

Let us outline the organization of this article. In Section 1 we recall the definition of Lie-Rinehart pairs, their universal enveloping algebras and their cohomology theory. In Sections 2 and 3 we describe the module structure on $H^\bullet(S, M)$ and present the spectral sequence. After proving some useful lemmas regarding eulerian modules in Section 4 we devote Section 5 to the computation of the Hochschild cohomology of the algebra of differential operators of a central arrangement of three lines. Finally, in Section 6 we provide the two other applications described above.

We will denote the tensor product over the base field \mathbb{k} simply by \otimes or, sometimes, by $|$. Unless it is otherwise specified, all vector spaces and algebras will be over \mathbb{k} . Given an associative algebra A , the enveloping algebra A^e is the vector space $A \otimes A$ endowed with the product \cdot defined by $a_1 \otimes a_2 \cdot b_1 \otimes b_2 = a_1 b_1 \otimes b_2 a_2$, so that the category of A^e -modules is equivalent to that of A -bimodules. The Hochschild cohomology of A with values on an A^e -module M is defined as $\text{Ext}_{A^e}^\bullet(A, M)$ and will be denoted by $H^\bullet(A, M)$ or, if $M = A$, by $HH^\bullet(A)$. The book [22] by C. Weibel may serve as general reference on this subject.

The first author heartfully thanks his PhD advisor M. Suárez-Álvarez for his collaboration, fruitful suggestions and overall help. We thank the Université Clermont Auvergne for hosting the first author in a postdoctoral position at the Laboratoire de Mathématiques Blaise Pascal during the year 2019-2020. Part of this work was done during the time the first author was supported by a full doctoral grant by CONICET and by the projects PIP-CONICET 12-20150100483, PICT 2015-0366 and UBA-CyT 20020170100613BA.

1. LIE-RINEHART ALGEBRAS

We begin by recalling some basic facts about Lie-Rinehart algebras available in [16] and in [8]. Until Section 3 we assume \mathbb{k} to be a field of arbitrary characteristic.

Definition 1.1. Let S and $(L, [-, -])$ be a commutative and a Lie algebra endowed with a morphism of Lie algebras $L \rightarrow \text{Der}_{\mathbb{k}}(S)$ that we write $\alpha \mapsto \alpha_S$ and a left S -module structure on L which we simply denote by juxtaposition. The pair (S, L) is a Lie–Rinehart algebra if the equalities

$$(\alpha_S)_S(t) = \alpha_S(t), \quad [\alpha, s\beta] = s[\alpha, \beta] + \alpha_S(s)\beta$$

hold whenever $s, t \in S$ and $\alpha, \beta \in L$.

Definition 1.2. Let (S, L) be a Lie–Rinehart algebra. A Lie–Rinehart module —or (S, L) -module— is a vector space M that is at the same time an S -module and an L -Lie module in such a way that

$$(\alpha_S) \cdot m = s \cdot (\alpha \cdot m), \quad \alpha \cdot (s \cdot m) = (\alpha_S) \cdot m + \alpha_S(s) \cdot m \quad (1)$$

for $s \in S$, $\alpha \in L$ and $m \in M$.

Theorem 1.3. Let (S, L) be a Lie–Rinehart algebra.

- (i) There exists an associative algebra $U = U(S, L)$, the universal enveloping algebra of (S, L) , endowed with a morphism of algebras $i : S \rightarrow U$ and a morphism of Lie algebras $j : L \rightarrow U$ that satisfies, for $s \in S$ and $\alpha \in L$,

$$i(s)j(\alpha) = j(\alpha_S), \quad j(\alpha)i(s) - i(s)j(\alpha) = i(\alpha_S(s)) \quad (2)$$

and universal with these properties.

- (ii) The category of U -modules is isomorphic to that of (S, L) -modules.

Example 1.4. The obvious actions of S and L make of S an U -module. If \mathfrak{g} is a Lie algebra then $(\mathbb{k}, \mathfrak{g})$ is a Lie–Rinehart algebra whose enveloping algebra is simply the usual enveloping algebra of \mathfrak{g} . If $S = \mathbb{k}[x_1, \dots, x_n]$ then the full Lie algebra of derivations $L = \text{Der}_{\mathbb{k}} S$ is a Lie–Rinehart algebra and its enveloping algebra is isomorphic to the algebra of differential operators $\text{Diff}(S) = A_n$, the n th Weyl algebra.

Definition 1.5. Let (S, L) be a Lie–Rinehart algebra with enveloping algebra U and let N be an U -module. The Lie–Rinehart cohomology of (S, L) with values on N is

$$H_S^\bullet(L, N) := \text{Ext}_U^\bullet(S, N).$$

In many important situations, some of which will be illustrated in the examples below, L is a projective S -module, and in this case there is a well-known complex that computes the Lie–Rinehart cohomology.

Proposition 1.6. Suppose that L is S -projective and let $\Lambda_S^\bullet L$ denote the exterior algebra of L over S . The complex $\text{Hom}_S(\Lambda_S^\bullet L, N)$ with Chevalley–Eilenberg differentials computes $H_S^\bullet(L, N)$.

Example 1.7. For the Lie–Rinehart algebra $(\mathbb{k}, \mathfrak{g})$ with \mathfrak{g} a Lie algebra, N is simply a \mathfrak{g} -Lie module and the complex $\text{Hom}_{\mathbb{k}}(\Lambda_{\mathbb{k}}^\bullet L, N)$ is the standard complex that computes the Lie algebra cohomology $H^\bullet(\mathfrak{g}, N)$.

Given a finite dimensional manifold M , we obtain a Lie–Rinehart algebra setting $S = C^\infty(M)$, the algebra of smooth functions, and $L = \mathfrak{X}(M)$, the Lie algebra of vector fields on M . The enveloping algebra of this pair is isomorphic to the algebra of globally defined differential operators on the manifold —see [8, §1]. We can find in J. Nestruev’s [13, Proposition 11.32] that L is finitely generated and projective over S ; as the complex $\mathrm{Hom}_S(\Lambda_S^\bullet L, S)$ is the de Rham complex $\Omega^\bullet(M)$ of differential forms, the cohomology $H_S^\bullet(L, S)$ coincides with the de Rham cohomology of M .

Example 1.8. A central *hyperplane arrangement* \mathcal{A} in a finite dimensional vector space V is a finite set $\{H_1, \dots, H_l\}$ of subspaces of codimension 1. Let $\lambda_i : V \rightarrow \mathbb{k}$ be a linear form with kernel H_i for each $i \in \{1, \dots, l\}$. We let S be the algebra of polynomial functions on V , fix a defining polynomial $Q = \lambda_1 \cdots \lambda_l \in S$ for \mathcal{A} and consider the Lie algebra

$$\mathrm{Der} \mathcal{A} := \{\theta \in \mathrm{Der}_{\mathbb{k}}(S) : Q \text{ divides } \theta(Q)\}$$

of derivations tangent to the arrangement. The pair $(S, \mathrm{Der} \mathcal{A})$ is a Lie–Rinehart algebra, as one can readily check.

An arrangement \mathcal{A} is *free*, by definition, if $\mathrm{Der} \mathcal{A}$ is a free S -module. In that case, as in [12, Theorem 1.3.1], the enveloping algebra of $(S, \mathrm{Der} \mathcal{A})$ is isomorphic to the *algebra of differential operators tangent to the arrangement* $\mathrm{Diff} \mathcal{A}$, that is, the algebra of differential operators on S which preserve the ideal QS of S and all its powers. As seen in [5] or by M. Suárez-Álvarez in [19], it coincides with the associative algebra generated inside the algebra $\mathrm{End}_{\mathbb{k}}(S)$ of linear endomorphisms of the vector space S by $\mathrm{Der} \mathcal{A}$ and the set of maps given by left multiplication by elements of S .

For the Lie–Rinehart algebra (S, L) associated to a free hyperplane arrangement \mathcal{A} , the complex $\mathrm{Hom}_S(\Lambda_S^\bullet L, S)$ is the complex of logarithmic forms $\Omega^\bullet(\mathcal{A})$, and its cohomology is isomorphic to the Orlik–Solomon algebra of \mathcal{A} —here we refer to J. Wiens and S. Yuzvinsky’s [23]. When $\mathbb{k} = \mathbb{C}$, this algebra is, in turn, isomorphic to the cohomology of the complement of the arrangement, as proved by P. Orlik and L. Solomon in [14].

2. THE U -MODULE STRUCTURE ON $H^\bullet(S, M)$

Let (S, L) be a Lie–Rinehart algebra such that L is a projective S -module. Let U be its enveloping algebra and M be an U^e -module. Since the inclusion of S in U is a morphism of algebras we can regard M as an S^e -module and consider the Hochschild cohomology of S with values on M , denoted as before by $H^\bullet(S, M)$. In this section we first construct an U -module structure on $H^\bullet(S, M)$ from an U^e -injective resolution of M ; afterwards, we construct S - and L -module structures on $H^\bullet(S, M)$ from an S^e -projective resolution of S ; finally, we show that these induce an U -module structure that coincides with the one we have using injectives: this will allow us to compute the latter in practice.

2.1. Using U^e -injective modules. The second author and P. Le Meur introduce in [10, Lemma 3.2.1] the functor

$$G = \mathrm{Hom}_{S^e}(S, -) : U^e \mathrm{Mod} \rightarrow U \mathrm{Mod}, \quad (3)$$

where for an U^e -module M the left L -Lie module and left S -module structures on $\text{Hom}_{S^e}(S, M)$ are defined by the rules

$$\begin{aligned} (\alpha \cdot \varphi)(s) &= (\alpha \otimes 1) \cdot \varphi(s) - (1 \otimes \alpha) \cdot \varphi(s) - \varphi(\alpha_S(s)), \\ (t \cdot \varphi)(s) &= (t \otimes 1) \cdot \varphi(s) \end{aligned} \tag{4}$$

for $\alpha \in L$, $\varphi \in \text{Hom}_{S^e}(S, M)$ and $s, t \in S$.

Proposition 2.1. *Let M be an U^e -module and let $M \rightarrow I^\bullet$ be an injective resolution of M as an U^e -module. The cohomology of the complex $G(I^\bullet) = \text{Hom}_{S^e}(S, I^\bullet)$ is the Hochschild cohomology $H^\bullet(S, M)$.*

Proof. Let I be an injective U^e -module. The functor $\text{Hom}_{S^e}(-, I)$ is naturally isomorphic to $\text{Hom}_{U^e}(U^e \otimes_{S^e} -, I)$, which is the composition of the exact functor $\text{Hom}_{U^e}(-, I)$ and $U^e \otimes_{S^e} -$. Now, the PBW-theorem in [16, §3] ensures that U is a projective S -module and, using Proposition IX.2.3 of H. Cartan and S. Eilenberg's [6], we obtain that U^e is S^e -projective. As a consequence of this, the functor $U^e \otimes_{S^e} -$ is exact and therefore $\text{Hom}_{S^e}(-, I)$ is exact as well. This implies that $M \rightarrow I^\bullet$ is in fact a resolution of M by S^e -injective modules, so that $H^\bullet(\text{Hom}_{S^e}(S, I^\bullet)) = \text{Ext}_{S^e}(S, M)$. \square

From Proposition 2.1 and the functoriality of $G = \text{Hom}_{S^e}(S, -)$ we can conclude that if $M \rightarrow I^\bullet$ is an U^e -injective resolution then the U -module structure on $\text{Hom}_{S^e}(S, I^\bullet)$ defined in (4) induces an U -module structure on $H^\bullet(S, M)$:

Corollary 2.2. *Let M be an U^e -module and let $M \rightarrow I^\bullet$ be an U^e -injective resolution. Let $j \geq 0$, $u \in U$ and denote the class in $H^j(S, M)$ of $\varphi \in \text{Hom}_{S^e}(S, I^j)$ by $\bar{\varphi}$. Defining $u \cdot \bar{\varphi}$ to be the class of $u \cdot \varphi$ as defined in (4) we obtain an U -module structure on $H^j(S, M)$.*

2.2. Using S^e -projective modules. In this subsection we define S - and L -module structures on $H^\bullet(S, M)$ using projectives. To see that these structures are compatible as in (1) we will show that they are equal to the ones in Subsection 2.1 using injectives and conclude that they determine an U -module structure.

2.2.1. The S -module structure. We start by letting $P_\bullet \rightarrow S$ be an S^e -projective resolution. For each $i \geq 0$ there is a left S -module structure on $\text{Hom}_{S^e}(P_i, S)$ given by

$$(s \cdot \phi)(p) = s\phi(p) \quad \text{for } s \in S, \phi \in \text{Hom}_{S^e}(P_i, S) \text{ and } p \in P_i. \tag{5}$$

With this structure the differentials in the complex $\text{Hom}_{S^e}(P_\bullet, S)$ become S -linear and therefore the cohomology of this complex, which is canonically isomorphic to $H^\bullet(S, M)$, inherits an S -module structure. It is straightforward to verify that this structure does not depend on the choice of the projective resolution.

2.2.2. δ -liftings. To give an L -Lie module structure on $H^\bullet(S, M)$ using projectives we will use the tools developed by M. Suárez-Álvarez in [20]. Let A be an algebra and $\delta : A \rightarrow A$ a derivation. Given an A -module V , we say that a linear map $f : V \rightarrow V$ is a δ -operator if for every $a \in A$ and $v \in V$ we have

$$f(av) = \delta(a)v + af(v).$$

If, moreover, $\varepsilon : P_\bullet \rightarrow V$ is an A -projective resolution of V , a δ -lifting of f to P_\bullet is a family of δ -operators $f_\bullet = (f_i : P_i \rightarrow P_i, i \geq 0)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & V \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & V \end{array}$$

The construction in [20, §1] proceeds then as follows. Given an algebra A with a derivation δ , a δ -operator $f : V \rightarrow V$ and a projective resolution $P_\bullet \rightarrow V$, a δ -lifting f_\bullet of f to P_\bullet is shown to always exist. This δ -lifting gives rise to an endomorphism f_\bullet^\sharp of the complex $\text{Hom}_A(P_\bullet, V)$ defined for $i \geq 0$ and $\varphi \in \text{Hom}_A(P_i, V)$ by $f_i^\sharp(\varphi) = f \circ \varphi - \varphi \circ f_i$. Moreover, f_\bullet^\sharp induces an endomorphism ∇_f^\bullet of the cohomology $\text{Ext}_A^\bullet(V, V)$ which, conveniently, does not depend neither on the choice of the δ -lifting or the projective resolution.

We will now generalize this construction so that we can adapt it to our needs. Let us first recall two simple but fundamental results in the following Lemma.

Lemma 2.3 ([20, §1.4, §1.6]). *Let V be a left A -module, let $f : V \rightarrow V$ be a δ -operator and let $\varepsilon : P_\bullet \rightarrow V$ be a projective resolution.*

- (i) *There exists a δ -lifting f_\bullet of f to P_\bullet .*
- (ii) *If $\varepsilon' : P'_\bullet \rightarrow V$ is another projective resolution, f_\bullet and f'_\bullet are δ -liftings of f to ε and ε' and $h_\bullet : P'_\bullet \rightarrow P_\bullet$ is an A -linear lifting of $\text{id}_V : V \rightarrow V$ then $f_\bullet h_\bullet - h_\bullet f'_\bullet : P'_\bullet \rightarrow P_\bullet$ is an A -linear lifting of the zero map $0 : V \rightarrow V$.*

Proposition 2.4. *Let V and W be two A -modules, $f : V \rightarrow V$ and $g : W \rightarrow W$ two δ -operators and $P_\bullet \rightarrow V$ an A -projective resolution. Let $f_\bullet = (f_i)_{i \geq 0}$ be a δ -lifting of f to P_\bullet provided by Proposition 2.3.*

- (i) *There is an endomorphism $(f_\bullet, g) = ((f_i, g))_{i \geq 0}$ of the complex of vector spaces $\text{Hom}_A(P_\bullet, W)$ such that if $i \geq 0$ and $\phi \in \text{Hom}_A(P_i, W)$ then*

$$(f_i, g)(\phi) = g \circ \phi - \phi \circ f_i. \tag{6}$$

- (ii) *The map $\nabla_{(f_\bullet, g)}^\bullet : \text{Ext}_A^\bullet(V, W) \rightarrow \text{Ext}_A^\bullet(V, W)$ induced by (f_\bullet, g) in cohomology is independent of the choice of the projective resolution $P_\bullet \rightarrow V$ and the δ -lifting f_\bullet .*

Proof. Let $i \geq 0$. As both $g : W \rightarrow W$ and $f_i : P_i \rightarrow P_i$ are δ -operators and ϕ is A -linear, the difference $(f_i, g)(\phi) = g \circ \phi - \phi \circ f_i$ is A -linear. That (f_\bullet, g) is a morphism of complexes is an immediate consequence of the fact that so is f_\bullet .

For the second assertion we let $\varepsilon' : P'_\bullet \rightarrow V$ be another A -projective resolution of V , f'_\bullet be another δ -lifting of f to P_\bullet and (f'_\bullet, g) be the graded endomorphism of $\text{Hom}_A(P'_\bullet, W)$ in (6). We claim that if $h : P'_\bullet \rightarrow P_\bullet$ is a morphism of complexes lifting the identity of S then the diagram

$$\begin{array}{ccc} \text{Hom}_{S^e}(P_\bullet, W) & \xrightarrow{(f_\bullet, g)} & \text{Hom}_{S^e}(P_\bullet, W) \\ \downarrow h_\bullet^* & & \downarrow h_\bullet^* \\ \text{Hom}_{S^e}(P'_\bullet, W) & \xrightarrow{(f'_\bullet, g)} & \text{Hom}_{S^e}(P'_\bullet, W) \end{array} \quad (7)$$

commutes up to homotopy.

Proposition 2.3 tells us that $z_\bullet := f_\bullet h_\bullet - h_\bullet f'_\bullet : P'_\bullet \rightarrow P_\bullet$ is an A -linear lifting of $0 : V \rightarrow V$ and therefore $z_\bullet^* : \text{Hom}_A(P_\bullet, W) \rightarrow \text{Hom}_A(P'_\bullet, W)$ is homotopic to zero. To prove the claim it is then enough to show that

$$(f'_i, g) \circ h_i^* - h_i^* \circ (f_i, g) = z_i^* \quad \text{for each } i \geq 0,$$

so that the zero-homotopic map z_\bullet^* is the failure in the commutativity of the diagram (7). We have, for $\phi \in \text{Hom}_{S^e}(P_i, W)$,

$$\begin{aligned} ((f'_i, g) \circ h_i^* - h_i^* \circ (f_i, g))(\phi) &= (f'_i, g)(\phi \circ h_i) - h_i^*((f_i, g)(\phi)) \\ &= g \circ (\phi \circ h_i) - (\phi \circ h_i) \circ f'_i - (g \circ \phi) \circ h_i + (\phi \circ f_i) \circ h_i \\ &= \phi \circ f_i \circ h_i - \phi \circ h_i \circ f'_i \\ &= (h_i^* f_i^* - f_i'^* h_i^*)(\phi) = z_i^*(\phi). \end{aligned}$$

This proves the claim, and it follows at once that the endomorphisms that (f_\bullet, g) and (f'_\bullet, g) induce on $\text{Ext}_A^\bullet(V, W)$ are equal. \square

2.2.3. The L -Lie module structure. Let (S, L) be a Lie–Rinehart algebra, M be an U^e -module and $\alpha \in L$. To adapt the construction of Subsection 2.2.2 to our situation we recall that α acts on S by the derivation $\alpha_S : S \rightarrow S$ and consider the following assertions.

- (i) The map $\alpha_S^e = \alpha_S \otimes 1 + 1 \otimes \alpha_S : S^e \rightarrow S^e$ is a derivation.
- (ii) Viewing S as an S^e -module via $(s_1 \otimes s_2) \cdot t := s_1 t s_2$, the derivation $\alpha_S : S \rightarrow S$ becomes an α_S^e -operator.
- (iii) The map $\alpha_M : M \rightarrow M$ such that $\alpha_M(m) = (\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot m$ satisfies

$$\alpha_M((s \otimes t) \cdot m) = \alpha_S^e(s \otimes t) \cdot m + (s \otimes t) \cdot \alpha_M(m) \quad \text{for } s, t \in S \text{ and } m \in M,$$

which is to say that, regarding M as an S^e -module, α_M is an α_S^e -operator.

The first two claims can be proved with a straightforward calculation; for the third one, we let α, s, t and m as before and see that

$$\begin{aligned} \alpha_M((s \otimes t) \cdot m) &= ((\alpha \otimes 1 - 1 \otimes \alpha)(s \otimes t)) \cdot m \\ &= (\alpha s \otimes t - s \otimes t \alpha) \cdot m = ((\alpha(s) + s \alpha) \otimes t - s \otimes (\alpha t - \alpha(t))) \cdot m \end{aligned}$$

$$= \alpha^e(s \otimes t) \cdot m + (s\alpha \otimes t - s \otimes \alpha t) \cdot m = \alpha^e(s \otimes t) \cdot m + s \otimes t \cdot \alpha_M(m)$$

since $\alpha_S(s) = s\alpha - \alpha s$, as in (2).

We may now specialize Proposition 2.4 to our situation. We take

$$\begin{aligned} A &= S^e, & \delta &= \alpha_S^e : S^e \rightarrow S^e, & V &= S, \\ f &= \alpha_S : S \rightarrow S, & W &= M, & g &= \alpha_M : M \rightarrow M \end{aligned}$$

and from this we obtain the maps $\alpha_\bullet^\sharp := (f_\bullet^\sharp, g)$ and $\nabla_\alpha^\bullet := \nabla_{(f, g)}^\bullet$. More concretely:

Remark 2.5. Let $\alpha \in L$, M an U^e -module and $\varepsilon : P_\bullet \rightarrow S$ an S^e -projective resolution. Let α_\bullet be an α_S^e -lifting of $\alpha_S : S \rightarrow S$ to P_\bullet , that is, a morphism of complexes $\alpha_\bullet = (\alpha_q : P_q \rightarrow P_q)_{q \geq 0}$ such that $\varepsilon \circ \alpha_0 = \alpha_S \circ \varepsilon$ and for each $q \geq 0$, $s, t \in S$ and $p \in P_q$

$$\alpha_q((s \otimes t) \cdot p) = (\alpha_S(s) \otimes t + s \otimes \alpha_S(t)) \cdot p + (s \otimes t) \cdot p.$$

Denote by $\alpha \otimes 1 - 1 \otimes \alpha : M \rightarrow M$ the map such that $m \mapsto (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m$. The endomorphism α_\bullet^\sharp of $\text{Hom}_{S^e}(P_\bullet, M)$ is given for each $q \geq 0$ by

$$\alpha_q^\sharp(\phi) = (\alpha \otimes 1 - 1 \otimes \alpha) \circ \phi - \phi \circ \alpha_q, \quad (8)$$

and the map $\nabla_\alpha^\bullet : H^\bullet(S, M) \rightarrow H^\bullet(S, M)$ is the unique graded endomorphism such that

$$\nabla_\alpha^q([\phi]) = [\alpha_q^\sharp(\phi)], \quad (9)$$

where $[-]$ denotes class in cohomology.

Proposition 2.6. *Let $\text{End}(H^\bullet(S, M))$ be the Lie algebra of linear endomorphisms of $H^\bullet(S, M)$ with Lie structure given by the commutator. The map $\nabla : L \rightarrow \text{End}(H^\bullet(S, M))$ defined by $\alpha \mapsto \nabla_\alpha^\bullet$ is a morphism of Lie algebras.*

Proof. Let $\alpha, \beta \in L$ and call $\gamma = [\alpha, \beta]$. Let $\alpha_\bullet, \beta_\bullet$ and γ_\bullet be α^e, β^e and γ_S^e -liftings, respectively. Observe that γ_\bullet is not necessarily the commutator of α_\bullet and β_\bullet . Let $\alpha_\bullet^\sharp, \beta_\bullet^\sharp$ and γ_\bullet^\sharp be the endomorphisms of $\text{Hom}_{S^e}(P_\bullet, M)$ defined as in (8) and consider the endomorphism θ^\bullet of $\text{Hom}_{S^e}(P_\bullet, M)$ such that if $i \geq 0$ and $\phi \in \text{Hom}_{S^e}(P_i, M)$

$$\theta^i(\phi) = (\gamma \otimes 1 - 1 \otimes \gamma) \circ \phi - \phi \circ (\alpha_i \circ \beta_i - \beta_i \circ \alpha_i).$$

A straightforward calculation shows that the commutator $\alpha_\bullet \circ \beta_\bullet - \beta_\bullet \circ \alpha_\bullet$ is a γ_S^e -lifting of γ and therefore Proposition 2.4 tells us that θ^\bullet and γ_\bullet^\sharp induce the same endomorphism on cohomology. We claim that in fact $\theta^\bullet = \alpha_\bullet^\sharp \circ \beta_\bullet^\sharp - \beta_\bullet^\sharp \circ \alpha_\bullet^\sharp$. Indeed, for $i \geq 0$ and $\phi \in \text{Hom}_{S^e}(P_i, M)$

$$\begin{aligned} \alpha_i^\sharp(\beta_i^\sharp(\phi)) &= (\alpha \otimes 1 - 1 \otimes \alpha) \circ \beta_i^\sharp(\phi) - \beta_i^\sharp(\phi) \circ \alpha_i \\ &= (\alpha \otimes 1 - 1 \otimes \alpha) \circ ((\beta \otimes 1 - 1 \otimes \beta) \circ \phi - \phi \circ \beta_i) \\ &\quad - (\beta \otimes 1 - 1 \otimes \beta) \circ \phi \circ \alpha_i - \phi \circ \beta_i \circ \alpha_i \\ &= (\alpha\beta \otimes 1 - \alpha \otimes \beta - \beta \otimes \alpha + \alpha \otimes \beta) \circ \phi - (\alpha \otimes 1 - 1 \otimes \alpha) \circ \phi \circ \alpha_i \\ &\quad - (\beta \otimes 1 - 1 \otimes \beta) \circ \phi \circ \alpha_i - \phi \circ \beta_i \circ \alpha_i \end{aligned}$$

These two expressions together with the equality $\alpha\beta - \beta\alpha = \gamma$ in U allow us to conclude that $\alpha_i^\sharp(\beta_i^\sharp(\phi)) - \beta_i^\sharp(\alpha_i^\sharp(\phi)) = \theta^i(\phi)$, which proves the claim.

We conclude in this way that

$$\begin{aligned} H^\bullet(\gamma^\sharp) &= H^\bullet(\theta^\bullet) = H^\bullet(\alpha_\bullet^\sharp \circ \beta_\bullet^\sharp - \beta_\bullet^\sharp \circ \alpha_\bullet^\sharp) \\ &= H^\bullet(\alpha_\bullet^\sharp) \circ H^\bullet(\beta_\bullet^\sharp) - H^\bullet(\beta_\bullet^\sharp) \circ H^\bullet(\alpha_\bullet^\sharp), \end{aligned}$$

in virtue of the linearity of the functor H . This means that $\nabla_\gamma^\bullet = [\nabla_\alpha^\bullet, \nabla_\beta^\bullet]$. \square

Example 2.7. It is easy to describe the endomorphism ∇_α^0 of $H^0(S, U)$ for a given $\alpha \in L$. Let us choose a resolution P_\bullet of S with $P_0 = S^e$ and augmentation $\varepsilon : S^e \rightarrow S$ defined by $\varepsilon(s \otimes t) = st$. As α_S^e is a α_S^e -operator and $\varepsilon \circ \alpha_S^e = \alpha_S \circ \varepsilon$, we may choose an α_S^e -lifting with $\alpha_0 = \alpha_S^e$. According to the rule (8) we have

$$\alpha_0^\sharp(\phi)(1 \otimes 1) = (\alpha \otimes 1 - 1 \otimes \alpha) \cdot \phi(1 \otimes 1) \quad \text{for all } \phi \in \text{Hom}_{S^e}(P_0, M). \quad (10)$$

Identifying, as usual, each $\phi \in \text{Hom}_{S^e}(S^e, U)$ with $\phi(1 \otimes 1) \in U$, we can view $H^0(S, U)$ as a subspace of U and then (10) tells us that $\nabla_\alpha^0(u) = \alpha u - u\alpha$ for all $u \in H^0(S, U)$.

2.3. Comparing the two actions. We now prove that the S - and L -module structures on $H^\bullet(S, M)$ constructed in Subsection 2.2 using projectives are equal to those induced by the U -module structure in Subsection 2.1 using injectives. As a consequence, this shows that the actions of S and L using projectives satisfy compatibility relations (1).

Theorem 2.8. *Suppose L is S -projective. The S - and L -module structures on $H^\bullet(S, M)$ determined by (4) using U^e -injective modules are equal to those given in (5) and (9) using S^e -projective modules.*

Proof. We will only prove that the L -module structures coincide—that the S -module structures are equal too is analogous and simpler. To begin with, we fix an U^e -injective resolution $\eta : M \rightarrow I^\bullet$, an S^e -projective resolution $\varepsilon : P_\bullet \rightarrow S$ and $\alpha \in L$. In (8), we give endomorphisms of complexes α_i^\sharp of $\text{Hom}_{S^e}(P_\bullet, M)$ and of $\text{Hom}_{S^e}(P_\bullet, I^j)$ for each $j \geq 0$ —we denote them the same way—which induce the map ∇_α^\bullet on their cohomologies $H^\bullet(S, M)$ and $H^\bullet(S, I^j)$. We first claim that the map

$$\eta_* : \text{Hom}_{S^e}(P_\bullet, M) \ni \phi \mapsto \eta \circ \phi \in \text{Hom}_{S^e}(P_\bullet, I^\bullet)$$

satisfies, for each $i \geq 0$ and $\phi \in \text{Hom}_{S^e}(P_i, M)$,

$$\eta_*(\alpha_i^\sharp(\phi)) = \alpha_i^\sharp(\eta_*(\phi)). \quad (11)$$

Indeed, since η is a morphism of U^e -modules it commutes with $1 \otimes \alpha - \alpha \otimes 1$ and thus

$$\begin{aligned} \eta_*(\alpha_i^\sharp(\phi)) &= \eta \circ (\alpha \otimes 1 - 1 \otimes \alpha) \circ \phi - \eta \circ \phi \circ \alpha_i \\ &= (\alpha \otimes 1 - 1 \otimes \alpha) \circ \eta \circ \phi - \eta \circ \phi \circ \alpha_i = \alpha_i^\sharp(\eta_*(\phi)). \end{aligned}$$

Let us see that, on the other hand, the map

$$\varepsilon^* : \text{Hom}_{S^e}(S, I^\bullet) \ni \varphi \mapsto \varphi \circ \varepsilon \in \text{Hom}_{S^e}(P_\bullet, I^\bullet)$$

satisfies that for each $\varphi \in \text{Hom}_{S^e}(S, I^\bullet)$

$$\varepsilon^*(\alpha \cdot \varphi) = \alpha_0^\sharp(\varepsilon^*(\varphi)). \quad (12)$$

Since α_\bullet is a lifting of $\alpha_S : S \rightarrow S$ to P_\bullet , we have that $\alpha \circ \varepsilon = \varepsilon \circ \alpha_0$ and

$$\begin{aligned} \varepsilon^*(\alpha \cdot \varphi) &= (\alpha \otimes 1 - 1 \otimes \alpha) \circ \varphi \circ \varepsilon - \varphi \circ \alpha \circ \varepsilon \\ &= (\alpha \otimes 1 - 1 \otimes \alpha) \circ \varphi \circ \varepsilon - \varphi \circ \varepsilon \circ \alpha_0 = \alpha_0^\sharp(\varepsilon^*(\varphi)). \end{aligned}$$

As the morphisms of complexes ε^* and η_* are quasi-isomorphisms, the fact that they are equivariant with respect to the actions of α —as shown by (11) and (12)—allows us to conclude that the two actions of L on $H^\bullet(S, M)$ coincide. \square

3. THE SPECTRAL SEQUENCE

Let (S, L) be a Lie–Rinehart algebra, let U be its enveloping algebra and let M be an U^e -module. In this section we construct a spectral sequence which converges to the Hochschild cohomology of U with values on M and whose second page involves the Lie–Rinehart cohomology of (S, L) and the Hochschild cohomology of S with values on M .

Recall that in (3) we considered a functor $G : U^e\text{Mod} \rightarrow U\text{Mod}$ defined on objects as $G(M) = \text{Hom}_{S^e}(S, M)$. We now consider the functor

$$\begin{aligned} F : U\text{Mod} &\rightarrow U^e\text{Mod} \\ F(N) &= U \otimes_S N \end{aligned} \quad (13)$$

where we give to $U \otimes_S N$ the U^e -module structure in [7, (2.4)]. This structure is completely determined by the rules

$$\begin{aligned} (v \otimes 1) \cdot u \otimes_S n &= vu \otimes_S n, \\ (1 \otimes \alpha) \cdot u \otimes_S n &= u\alpha \otimes_S n - u \otimes_S \alpha \cdot n, \quad (1 \otimes s) \cdot u \otimes_S n = u\alpha \otimes_S s \cdot n \end{aligned}$$

for $u, v \in U$, $n \in N$ and $\alpha \in L$. With the functors G and F at hand, we can state the very useful Proposition 3.4.1 of [10].

Proposition 3.1. *The functor F is left adjoint to G .*

Theorem 3.2. *Assume L is S -projective and let N and M be a left U -module and an U^e -module. There is a first-quadrant spectral sequence E_\bullet converging to $\text{Ext}_{U^e}^\bullet(F(N), M)$ with second page*

$$E_2^{p,q} = \text{Ext}_U^p(N, H^q(S, M)).$$

Proof. Let $Q_\bullet \rightarrow N$ be an U -projective resolution of N and let $M \rightarrow I^\bullet$ be an U^e -injective resolution. Consider the double complex

$$X^{\bullet,\bullet} = \text{Hom}_U(Q_\bullet, G(I^\bullet))$$

and denote its total complex by Z^\bullet . There are two spectral sequences for this double complex: we will use the first one to compute $H^\bullet(Z)$ and the second one will be the one we are looking for. From the first filtration on Z^\bullet with

$$\tilde{F}^q Z^p = \bigoplus_{\substack{r+s=p \\ s \geq q}} X^{r,s}$$

we obtain a first spectral sequence converging to $H(Z^\bullet)$. Its zeroth page \tilde{E}_0 is

$$\tilde{E}_0^{p,q} = \text{Hom}_U(Q_p, G(I^q))$$

and its differential comes from the one on Q_\bullet . We claim that for each $s \geq 0$, the functor $\text{Hom}_U(-, G(I^s))$ is exact. Indeed, by the adjunction of Proposition 3.1 it is naturally isomorphic to $\text{Hom}_{U^e}(F(-), I^s)$, which is the composition of the functors $F = U \otimes_S (-)$ and $\text{Hom}_{U^e}(-, I^s)$ and these are exact because U is left projective over S and I^s is U^e -injective. The first page \tilde{E}_1 of the spectral sequence is therefore given by

$$\tilde{E}_1^{p,q} = \begin{cases} \text{Hom}_U(N, G(I^q)) \cong \text{Hom}_{U^e}(F(N), I^q) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0 \end{cases}$$

and its differential is induced by that of I^\bullet . Now, as the complex $\text{Hom}_{U^e}(F(N), I^\bullet)$ computes $\text{Ext}_{U^e}^\bullet(F(N), M)$ using injectives, we obtain that the second page is

$$\tilde{E}_2^{p,q} = \begin{cases} \text{Ext}_{U^e}^q(F(N), M) & \text{if } p = 0; \\ 0 & \text{if } p \neq 0. \end{cases}$$

This spectral sequence thus degenerates at its the second page, so that we see that $H^\bullet(Z)$ is isomorphic to $\text{Ext}_{U^e}^\bullet(F(N), M)$.

The second filtration on Z^\bullet is given by

$$F^p Z^q = \bigoplus_{\substack{r+s=q \\ r \geq p}} X^{r,s}$$

and determines a second spectral sequence E_\bullet that also converges to $H(Z^\bullet)$. Its differential on E_0 is induced by the one on I^\bullet ; as Q_p is U -projective for each $p \geq 0$, the cohomology of $\text{Hom}_U(Q_p, G(I^\bullet))$ is given in its q th place precisely by $E_1^{p,q} = \text{Hom}_U(Q_p, H^q(S, M))$ —recall that, according to Proposition 2.1, the cohomology of $G(I^\bullet)$ is $H^\bullet(S, M)$. Since the differentials in E_1 are induced by those of Q_\bullet , for each $q \geq 0$ the cohomology of the row $E_1^{\bullet,q}$ is $E_2^{p,q} = \text{Ext}_U^p(N, H^q(S, M))$. The spectral sequence E_\bullet is therefore the one we were looking for. \square

Specializing Theorem 3.2 to the case in which $N = S$ we obtain the following corollary, which is in fact the result we are mainly interested in.

Corollary 3.3. *If L is S -projective then for each U^e -module M there is a first-quadrant spectral sequence E_\bullet converging to $H^\bullet(U, M)$ with second page*

$$E_2^{p,q} = H_S^p(L, H^q(S, M)).$$

The following examples illustrate what happens when we take $M = U$ in the two extreme situations.

Example 3.4. Suppose first that $L = 0$. The enveloping algebra U is just S and $\Lambda_S^\bullet L = S$, so the resolution $U \otimes \Lambda_S^\bullet L$ of S is simply $Q_\bullet = U \otimes_S S$. The double complex $X^{\bullet, \bullet}$ is therefore $\text{Hom}_S(S, \text{Hom}_{S^e}(S, I^\bullet))$, which is isomorphic to $\text{Hom}_{S^e}(S, I^\bullet)$ and the cohomology of the complex Z^\bullet in the proof is $HH^\bullet(S)$, the Hochschild cohomology of S .

Example 3.5. If $S = \mathbb{k}$ and $L = \mathfrak{g}$ is a Lie algebra then $H^\bullet(S, U) = \text{Ext}_{\mathbb{k}^e}^\bullet(\mathbb{k}, U)$ is just U , the second page of our spectral sequence is $H^\bullet(\mathfrak{g}, U)$ and we recover from Corollary 3.3 the well-known fact that the Hochschild cohomology of the enveloping algebra of a Lie algebra equals its Lie cohomology with values on U with the adjoint action, as in [6, XIII.5.1].

4. EULERIAN MODULES

We assume from now on that \mathbb{k} is a field of characteristic zero. In this section we pay attention to a particular but rather frequent situation in which some calculations to attain the second page of the spectral sequence in Corollary 3.3 can be significantly shortened. Let $S = \mathbb{k}[x_1, \dots, x_n]$. The usual graded algebra structure on S , such that $|x_i| = 1$ if $1 \leq i \leq n$, induces a grading on the Lie algebra $\text{Der } S$ that makes each partial derivative ∂_i have degree -1 . Let L be a Lie subalgebra of $\text{Der } S$ that is also an S -submodule of $\text{Der } S$ freely generated by homogeneous derivations $\alpha_1, \dots, \alpha_l$, where $\alpha_1 = e = x_1 \partial_1 + \dots + x_n \partial_n$ is the *eulerian* derivation. The pair (S, L) is a Lie-Rinehart algebra and, since L is free, its enveloping algebra U admits the set $\{\alpha_1^{n_1} \dots \alpha_l^{n_l} : n_1, \dots, n_l \geq 0\}$ as an S -module basis of U thanks to the PBW-theorem in [16, §3]. The graded structures on S and $\text{Der } S$ induce on L and on U a graded Lie algebra and a graded associative algebra structures.

Definition 4.1. A \mathbb{Z} -graded left U -module $N = \bigoplus_{i \in \mathbb{Z}} N_i$ is *eulerian* if the action of e on N satisfies $e \cdot n = in$ if $n \in N_i$.

4.1. The Lie–Rinehart cohomology $H_S(L, N)$. Recall from Proposition 1.6 that the Lie-Rinehart cohomology of (S, L) with values on an U -module N is the cohomology of the complex $C_S^\bullet(L, N) = \text{Hom}_S(\Lambda_S^\bullet L, N)$ with differentials $d^r : C_S^r(L, N) \rightarrow C_S^{r+1}(L, N)$ determined by

$$\begin{aligned} (d^r f)(\alpha_{i_1} \wedge \dots \wedge \alpha_{i_{r+1}}) &= \sum_{j=1}^{r+1} (-1)^{j+1} \alpha_{i_j} \cdot f(\alpha_{i_1} \wedge \dots \wedge \check{\alpha}_{i_j} \wedge \dots \wedge \alpha_{i_{r+1}}) \\ &+ \sum_{1 \leq j < k \leq r+1} (-1)^{j+k} f([\alpha_j, \alpha_k] \wedge \alpha_{i_1} \wedge \dots \wedge \check{\alpha}_{i_j} \wedge \dots \wedge \check{\alpha}_{i_k} \wedge \dots \wedge \alpha_{i_{r+1}}), \end{aligned}$$

with $f \in \text{Hom}_S(\Lambda_S^r L, N)$ and $1 \leq i_1 < \dots < i_{r+1} \leq l$ and where $\check{\alpha}_i$ means that α_i has been omitted. The gradings on S , L and N induce a grading on each of the vector

spaces in the complex $C_S^\bullet(L, N)$ and the differentials are homogeneous with respect with this grading, so that, if $C_S^\bullet(L, N)_i$ is the subcomplex of $C_S^\bullet(L, N)$ of degree i , there is a decomposition $C_S^\bullet(L, N) = \bigoplus_{i \in \mathbb{Z}} C_S^\bullet(L, N)_i$. The cohomology of $C_S^\bullet(L, N)$ is a graded complex: we write $H_S^p(L, N) = \bigoplus_{i \in \mathbb{Z}} H_S^p(L, N)_i$, with $H_S^p(L, N)_i = H^p(C_S^\bullet(L, N)_i)$ for each $p \geq 0$. The next proposition allows us to see that $H_S(L, N) = H_S(L, N)_0$.

Proposition 4.2. *Let N be an eulerian U -module. The inclusion of the component of degree zero $C_S^\bullet(L, N)_0 \hookrightarrow C_S^\bullet(L, N)$ is a quasi-isomorphism.*

Proof. Let $\gamma^\bullet : C_S^\bullet(L, N) \rightarrow C_S^\bullet(L, N)$ be the linear map whose restriction to each homogeneous component of the complex $C_S^\bullet(L, N)$ is the multiplication by degree. A straightforward calculation shows that the homotopy $s = (s_r : C_S^r(L, N) \rightarrow C_S^{r-1}(L, N))_{r \geq 0}$ given by $(s_r f)(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}) = f(e \wedge \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r})$ satisfies $s \circ d + d \circ s = \gamma$. We obtain from this that γ induces the zero map in cohomology and then, as the field has characteristic zero, each of the cohomologies of the subcomplexes of nonzero degree are trivial. \square

Corollary 4.3. *If N is an eulerian U -module then the subspace $\bigcap_{i \geq 2} \ker(\alpha_i : N_0 \rightarrow N)$ of N_0 is isomorphic to $H_S^0(L, N)$.*

4.2. The Hochschild cohomology $H^\bullet(S, M)$. To compute the Hochschild cohomology of S we use the Koszul resolution of S available in [22, §4.5].

Lemma 4.4. *Let W be the subspace of S with basis (x_1, \dots, x_n) . The complex $P_\bullet = S^e \otimes \Lambda^\bullet W$ with differentials $b_\bullet : P_\bullet \rightarrow P_{\bullet-1}$ defined for $s, t \in S$ and $1 \leq i_1 < \cdots < i_r \leq n$ by*

$$b_r(s|t \otimes x_{i_1} \wedge \cdots \wedge x_{i_r}) = \sum_{j=1}^r (-1)^{j+1} (s x_{i_j} | t - s | x_{i_j} t) \otimes x_{i_1} \wedge \cdots \wedge \check{x}_{i_j} \wedge \cdots \wedge x_{i_r}$$

and augmentation $\varepsilon : S^e \rightarrow S$ given by $\varepsilon(s|t) = st$ is a resolution of S by free S^e -modules. Here the symbol $|$ denotes the tensor product inside S^e and \check{x}_{i_j} means that x_{i_j} is omitted.

If M is an graded U^e -module M , the cohomology of the complex $\text{Hom}_{S^e}(P_\bullet, M)$ is $H^\bullet(S, M)$. The graded algebra S induces a grading on this complex which is preserved by the differentials and therefore $H^\bullet(S, M)$ inherits a graded structure. We denote by $H^\bullet(S, M)_i$ the i th homogeneous component of $H^\bullet(S, M)$ for each $i \in \mathbb{Z}$.

Proposition 4.5. *If $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a graded U^e -module such that $(e \otimes 1 - 1 \otimes e) \cdot m = im$ for all $m \in M_i$ then for each $q \in \mathbb{Z}$ the q th Hochschild cohomology space $H^q(S, M)$ is an eulerian U -module.*

Proof. That $H^q(S, U)_i$ is a graded U -module for each i can be seen from (4). Following Remark 2.5 we denote by $e_S : S \rightarrow S$ the action of e on S and by e_S^e the derivation $e_S \otimes 1 + 1 \otimes e_S : S^e \rightarrow S^e$. We let, for $q \geq 0$, $e_q : P_q \rightarrow P_q$ be the e_S^e -operator such that

$$e_q(1|1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_q}) = q|1 \otimes x_{i_1} \wedge \cdots \wedge x_{i_q} \quad (14)$$

if $1 \leq i_1 < \dots < i_q \leq n$. A small calculation allows us to deduce from (14) that the collection of maps $(e_q)_{q \geq 0}$ is a e_S^e -lifting of e_S to P_\bullet . Let now $\phi \in \text{Hom}_{S^e}(P_q, M)$ be an homogeneous map of degree i and write $m_{i_1, \dots, i_q} := \phi(1|1 \otimes x_{i_1} \wedge \dots \wedge x_{i_q}) \in M_{i+q}$. Our hypothesis on M allows us to see that

$$\begin{aligned} e_q^\sharp(\phi)(1|1 \otimes x_{i_1} \wedge \dots \wedge x_{i_q}) \\ &= (e \otimes 1 - 1 \otimes e) \cdot m_{i_1, \dots, i_q} - \phi \circ e_q(1|1 \otimes x_{i_1} \wedge \dots \wedge x_{i_q}) \\ &= (i + q)m_{i_1, \dots, i_q} - qm_{i_1, \dots, i_q} = im_{i_1, \dots, i_q} \end{aligned}$$

and therefore $\nabla_e^q([\phi]) = i[\phi]$. \square

5. THE ALGEBRA OF DIFFERENTIAL OPERATORS TANGENT TO A CENTRAL ARRANGEMENT OF THREE LINES

In this section we describe the example that motivated us to construct the spectral sequence of Corollary 3.3: it is the algebra of differential operators $\text{Diff}\mathcal{A}$ tangent to a central arrangement of lines \mathcal{A} , whose Hochschild cohomology was studied by the first author and M. Suárez-Álvarez in [9]. We will regard $\text{Diff}\mathcal{A}$ as the enveloping algebra of a Lie–Rinehart algebra and compute the second page $E_2^{p,q} = H_S^p(L, H^q(S, U))$ of the spectral sequence of Corollary 3.3 for a central line arrangement of three lines. After studying the Lie–Rinehart cohomology in a generic situation, we will compute what we need of $H^\bullet(S, U)$ and the action of U to obtain the second page and, finally, the Hochschild cohomology $HH^\bullet(U)$ in Corollary 5.9.

Let $S = \mathbb{k}[x, y]$ and write the defining polynomial of the arrangement $Q = xF$ with $F = y(tx + y)$, for some $t \in \mathbb{k}$. H. Saito's criterion [17, Theorem 1.8.ii] allows us to see that the two derivations

$$E = x\partial_x + y\partial_y, \quad D = F\partial_y$$

form an S -basis of $\text{Der}\mathcal{A}$. In [9] there is a convenient presentation of $U = \text{Diff}\mathcal{A}$. It is generated by the symbols x, y, D and E subject to the relations

$$\begin{aligned} [y, x] &= 0, \\ [D, x] &= 0, & [D, y] &= F, \\ [E, x] &= x, & [E, y] &= y, & [E, D] &= D, \end{aligned}$$

where the bracket $[a, b]$ between two elements stands for the commutator $ab - ba$. Moreover, the set $\{x^{i_1}y^{i_2}D^{i_3}E^{i_4} : i_1, \dots, i_4 \geq 0\}$ is a basis of U as a vector space.

As in Section 4, we view S as a graded algebra, with both x and y of degree 1, and for each $i \geq 0$ we write S_i the homogeneous component of S of degree i . This grading induces one in $L := \text{Der}\mathcal{A}$ and also on U :

Proposition 5.1. *There is a grading on the algebra U with $|x| = |y| = |D| = 1$ and $|E| = 0$. Given $i \geq 0$ the i th homogeneous component U_i of U is the right $\mathbb{k}[E]$ -module generated by the set $\{x^r y^s D^t : r + s + t = i\}$.*

For convenience, we denote by ψ' the image of $\psi \in \mathbb{k}[E]$ under the linear map $\mathbb{k}[E] \rightarrow \mathbb{k}[E]$ such that $E^n \mapsto E^n - (E+1)^n$ for every $n \geq 0$. Recall that \otimes or $|$ denote the tensor product over \mathbb{k} and that we may sometimes omit it to alleviate notation.

5.1. The Lie-Rinehart cohomology $H_S^\bullet(L, N)$. We let V_L be the subspace of L with basis (D, E) and V_L^* be its dual space, and denote the dual basis by (\hat{D}, \hat{E}) . Let N be an eulerian U -module. The Lie-Rinehart cohomology $H_S^\bullet(L, N)$ of (S, L) with values on N is the cohomology of the complex $C_S^\bullet(L, N)$, which is isomorphic via standard identifications to the complex $N \otimes \Lambda^\bullet V_L^*$ given by

$$N \xrightarrow{d^0} N \otimes V_L^* \xrightarrow{d^1} N \otimes \Lambda^2 V_L^*$$

with differentials

$$\begin{aligned} d^0(n) &= D \cdot n \otimes \hat{D} + E \cdot n \otimes \hat{E}; \\ d^1(n \otimes \hat{D} + m \otimes \hat{E}) &= (D \cdot m - E \cdot n + n) \otimes \hat{D} \wedge \hat{E}. \end{aligned}$$

Proposition 5.2. *Let $N = \bigoplus_{i \in \mathbb{Z}} N_i$ be an eulerian U -module and $\nabla_D : N_0 \rightarrow N_1$ be the restriction of the action of D . There are isomorphisms of vector spaces*

$$H_S^p(L, N) \cong \begin{cases} \ker \nabla_D, & \text{if } p = 0; \\ \text{coker } \nabla_D \otimes \mathbb{k}\hat{D} \oplus \ker \nabla_D \otimes \mathbb{k}\hat{E}, & \text{if } p = 1; \\ \text{coker } \nabla_D \otimes \mathbb{k}\hat{D} \wedge \hat{E}, & \text{if } p = 2 \end{cases}$$

and $H_S^p(L, N) = 0$ for every other $p \in \mathbb{Z}$.

We notice that the cohomology $H_S^\bullet(L, N)$ depends only on the map $N_0 \rightarrow N_1$ given by multiplication by D .

Proof. Thanks to Proposition 4.2, we need only compute the cohomology of the subcomplex of $N \otimes \Lambda^\bullet V_L^*$ of degree zero. This subcomplex is

$$N_0 \xrightarrow{d_0^0} N_1 \otimes \mathbb{k}\hat{D} \oplus N_0 \otimes \mathbb{k}\hat{E} \xrightarrow{d_0^1} N \otimes \mathbb{k}\hat{D} \wedge \hat{E}$$

with differentials given by $d_0^0(n) = D \cdot n \otimes \hat{D}$ and $d_0^1(n \otimes \hat{D} + m \otimes \hat{E}) = D \cdot m \otimes \hat{D} \wedge \hat{E}$. The claim in the proposition follows immediately from these expressions. \square

In Proposition 4.5 we saw that the U -modules $H^\bullet(S, U)$ are eulerian and, as a consequence of this, to get $H_S^\bullet(L, H^\bullet(S, U))$ we may use the following strategy: to compute the homogeneous components of degree 0 and 1 of $H^\bullet(S, U)$ and then to describe the map $\nabla_D^\bullet : H^\bullet(S, U)_0 \rightarrow H^\bullet(S, U)_1$ given by the action of D .

5.2. **The Hochschild cohomology $H^\bullet(S, U)$.** Let W be the subspace of S with basis (x, y) . Applying $\text{Hom}_{S^e}(-, U)$ to the Koszul resolution in Lemma 4.4 and using standard identifications we obtain the complex

$$U \xrightarrow{\delta^0} U \otimes \text{Hom}(W, \mathbb{k}) \xrightarrow{\delta^1} U \otimes \text{Hom}(\Lambda^2 W, \mathbb{k}) \quad (15)$$

with differentials

$$\begin{aligned} \delta^0(u) &= [x, u]\hat{x} + [y, u]\hat{y} \\ \delta^1(a\hat{x} + b\hat{y}) &= ([x, b] - [y, a])\hat{x} \wedge \hat{y}, \end{aligned}$$

where (\hat{x}, \hat{y}) is the dual basis of (x, y) and $\hat{x} \wedge \hat{y}$ is the linear morphism $\Lambda^2 W \rightarrow \mathbb{k}$ that sends $x \wedge y$ to one. The cohomology of the complex (15) is $H^\bullet(S, U)$.

Proposition 5.3. *There are isomorphisms of graded vector spaces $H^0(S, U) \cong S$ and $H^2(S, U) \cong \mathbb{k}[D] \otimes \mathbb{k}[E] \otimes \mathbb{k}(\hat{x} \wedge \hat{y})$.*

Proof. Evidently, $H^0(S, U)$, the subset of U of elements that commute with x and y , contains S : let us prove that they are equal. Given $u \in H^0(S, U)$, there exist v_0, \dots, v_m in the subalgebra of U generated by x, y and D such that $u = \sum_{i=0}^m v_i E^i$. The condition $0 = [u, x]$ implies that $0 = \sum_{i=0}^m v_i (E^i)'$ and therefore that $v_i = 0$ for every $i > 0$, so that there exist $f_1, \dots, f_n \in S$ such that $u = \sum_{i=0}^n f_i D^i$. An inductive argument using that

$$0 = [u, y] = \sum_{i=0}^n f_i [D^i, y] \equiv n f_n D^{n-1} \pmod{\bigoplus_{i=0}^{n-2} S D^i}$$

allows us to see that $f_i = 0$ if $i > 1$ and therefore to conclude that $u \in S$.

We compute $H^2(S, U)$ directly from the complex (15). Denote by $S_{\geq 1}$ the space of polynomials with no constant term. We claim that $S_{\geq 1} D^k \mathbb{k}[E]$ is contained in the image of δ^1 for every $k \geq 0$. Indeed, if $f, g \in S$ and $\psi \in \mathbb{k}[E]$ then

$$\delta^1(g\varphi\hat{x} + f\psi\hat{y}) = (xf\psi' - yg\varphi')\hat{x} \wedge \hat{y},$$

so that our claim is true if $k = 0$. Assume now that $k > 0$ and that for every $j < k$ the inclusion $S_{\geq 1} D^j \mathbb{k}[E] \subset \text{Im } \delta^1$ holds. Given $f \in S$ and $\psi \in \mathbb{k}[E]$, we have that

$$\delta^1(f D^k \psi \hat{y}) = x f D^k \psi' \hat{x} \wedge \hat{y}$$

and

$$\begin{aligned} \delta^1(f D^k \psi \hat{x}) &= (-f[y, D^k]\psi - f D^k y \psi') \hat{x} \wedge \hat{y} \\ &= (-f[y, D^k](\psi - \psi') - f y D^k \psi') \hat{x} \wedge \hat{y} \\ &\equiv -f y D^k \psi' \hat{x} \wedge \hat{y} \pmod{\text{Im } \delta^1}, \end{aligned}$$

which proves the claim. We easily see, on the other hand, that the intersection of $\mathbb{k}[D]\mathbb{k}[E]$ with $\text{Im } \delta^1$ is trivial, so that $H^2(S, U) \cong \mathbb{k}[D]\mathbb{k}[E]\hat{x} \wedge \hat{y}$, as we wanted. \square

The computation of $H^1(S, U)$ is significantly more involved than the one just above. As we are after the Lie–Rinehart cohomology $H_{\mathfrak{S}}^{\bullet}(L, H^1(S, U))$, thanks to Proposition 5.2 we need only compute the homogeneous components of $H^1(S, U)$ of degree 0 and 1.

Proposition 5.4. *The graded vector space $H^1(S, U)$ satisfies $\dim H^1(S, U)_0 = 5$ and $\dim H^1(S, U)_1 = 8$. Moreover, $H^1(S, U)_0$ is generated by the classes of the cocycles of the complex (15)*

$$\eta_1 = (-yE + D)\hat{x} + tyE\hat{y}, \quad \eta_2 = y\hat{x}, \quad \eta_3 = x\hat{y}, \quad \eta_4 = y\hat{y}, \quad \eta_5 = D\hat{y},$$

and $H^1(S, U)_1$ is generated by the classes of the cocycles

$$\begin{aligned} \zeta_1 &= (D^2 - 2yDE + y^2(E^2 - E))\hat{x} + (2tyDE + tFE + ty^2(E - E^2))\hat{y}, \\ \zeta_2 &= (-y^2E + yD)\hat{x} + ty^2E\hat{y}, \quad \zeta_3 = y^2\hat{x}, \quad \zeta_4 = x^2\hat{y}, \\ \zeta_5 &= xy\hat{y}, \quad \zeta_6 = xD\hat{y}, \quad \zeta_7 = yD\hat{y}, \quad \zeta_8 = D^2\hat{y}. \end{aligned}$$

Proof. The homogeneous component of degree zero of the complex (15) is

$$U_0 \xrightarrow{\delta_0^0} U_1\hat{x} \oplus U_1\hat{y} \xrightarrow{\delta_0^1} U_2\hat{x} \wedge \hat{y}$$

with $U_0 = \mathbb{k}[E]$, $U_1 = S_1\mathbb{k}[E] \oplus D\mathbb{k}[E]$,

$$U_2 = S_2\mathbb{k}[E] \oplus S_1D\mathbb{k}[E] \oplus D^2\mathbb{k}[E] \tag{16}$$

and differentials given by

$$\begin{aligned} \delta_0^0(\phi) &= x\phi'\hat{x} + y\phi'\hat{y}, \\ \delta_0^1((x\varphi_1 + y\varphi_2 + D\varphi_3)\hat{x}) &= (-xy\varphi'_1 - y^2\varphi'_2 - yD\varphi'_3 - F(\varphi'_3 - \varphi_3))\hat{x} \wedge \hat{y}, \\ \delta_0^1((x\psi_1 + y\psi_2 + D\psi_3)\hat{y}) &= (x^2\psi'_1 + xy\psi'_2 + xD\psi'_3)\hat{x} \wedge \hat{y}, \end{aligned}$$

where ϕ , φ 's and ψ 's denote elements of $\mathbb{k}[E]$.

Let $a, b \in U_1$ and let $\omega = a\hat{x} + b\hat{y}$ be a 1-cocycle. Up to adding a coboundary we may suppose that the component of a in $x\mathbb{k}[E]$ is zero: we may therefore write

$$a = y\varphi_2 + D\varphi_3, \quad b = x\psi_1 + y\psi_2 + D\psi_3, \tag{17}$$

with Greek letters in $\mathbb{k}[E]$. The coboundary $\delta_0^1(\omega)$ belongs to $U_2\hat{x} \wedge \hat{y}$, which decomposes as in (16). The vanishing of the component in $D^2\mathbb{k}[E]$ does not give any information, that of the one in $S_1D\mathbb{k}[E]$ tells us that $\varphi'_3 = \psi'_3 = 0$ and, finally, that of $S_2\mathbb{k}[E]$ tells us that

$$x^2\psi'_1 + xy\psi'_2 = y^2\varphi'_2 - F\varphi'_3. \tag{18}$$

Let us put $\lambda := \varphi_3$. Looking at the component on $y^2\mathbb{k}[E]$ of equation (18) and keeping in mind that $F = y^2 + tx$ we see that $\varphi'_2 = \lambda$ and, using this, that $x\psi'_1 + y\psi'_2 = -\lambda ty$. There exist then $\mu \in \mathbb{k}$ and $f_1 \in S_1$ such that

$$\varphi_2 = -\lambda E + \mu, \quad x\psi_1 + y\psi_2 = \lambda tyE + f.$$

As a cocycle $\omega = a\hat{x} + v\hat{y}$ satisfying (17) is a coboundary only if it is zero, we conclude that $H^1(S, U)_0 \cong \mathbb{k}\eta_1 \oplus \mathbb{k}y\hat{x} \oplus (S_1 \oplus \mathbb{k}D)\hat{y}$, with $\eta_1 = (-yE + D)\hat{x} + tyE\hat{y}$.

We now compute $H^1(S, U)_1$. The component of degree 1 of the complex (15) is

$$U_1 \xrightarrow{\delta_0^1} U_2\hat{x} \oplus U_2\hat{y} \xrightarrow{\delta_1^1} U_3\hat{x} \wedge \hat{y}$$

with $U_3 = S_3\mathbb{k}[E] \oplus S_2D\mathbb{k}[E] \oplus S_1D^2\mathbb{k}[E] \oplus D^3\mathbb{k}[E]$ and differentials

$$\begin{aligned} \delta_1^0(x\phi_1 + y\phi_2 + D\rho) &= (x^2\phi'_1 + xy\phi'_2 + xD\rho')\hat{x} + (xy\phi'_1 + y^2\phi'_2 + yD\rho' + F(\rho' - \rho))\hat{y}, \\ \delta_1^1\left(\left(\sum x^i y^j \varphi_{ij} + xD\varphi_1 + yD\varphi_2 + D^2\varphi\right)\hat{x}\right) &= -\sum x^i y^{j+1} \varphi'_{ij} - xyD\varphi'_1 - xF(\varphi'_1 - \varphi_1) - y^2D\varphi'_2 - yF(\varphi'_2 - \varphi_2) \\ &\quad - yD^2\varphi' - 2FD(\varphi'_2 - \varphi_2) - FF_y(\varphi' - \varphi), \\ \delta_1^1\left(\left(\sum x^i y^j \psi_{ij} + xD\psi_1 + yD\psi_2 + D^2\psi\right)\hat{y}\right) &= \sum x^{i+1} y^j \psi'_{ij} + x^2D\psi'_1 + xyD\psi'_2 + xD^2\psi'. \end{aligned}$$

In all the sums that appear here the indices i and j are such that $i + j = 2$ and we have omitted the factor $\hat{x} \wedge \hat{y}$ for δ_1^1 . Again, all Greek letters lie in $\mathbb{k}[E]$.

Let us put, once again, $\omega = a\hat{x} + b\hat{y}$, this time with a and b in U_2 . Up to coboundaries, we write, with the same conventions as before,

$$a = y^2\varphi_{02} + yD\varphi_2 + D^2\varphi, \quad b = \sum x^i y^j \psi_{ij} + xD\psi_1 + yD\psi_2 + D^2\psi.$$

Let us examine the condition $\delta_1^1(\omega) = 0$ component by component according to our description of U_2 in (16) above.

In $D^3\mathbb{k}[E]$ there is no condition at all. In $S_1D^2\mathbb{k}[E]$ we have $xD^2\psi' - yD^2\varphi' = 0$, so that ψ and φ are scalars. In $S_2D\mathbb{k}[E]$ the condition reads

$$x^2D\psi'_1 + xyD\psi'_2 = y^2D\varphi'_2 + 2FD(\varphi' - \varphi). \quad (19)$$

Writing $F = y^2 + txy$ and looking at the terms that are in $y^2\mathbb{k}[E]$ we find $0 = \varphi'_2 - 2\varphi$, and then $\varphi_2 = -2\varphi E + \lambda$ for some $\lambda \in \mathbb{k}$. What remains of (19) implies that $x\psi'_1 + y\psi'_2 = -2ty\varphi$ and therefore there exists $h \in S_1$ such that

$$xD\psi_1 + yD\psi_2 = 2\varphi tyDE + hD.$$

Finally, we look at $S_3\mathbb{k}[E]$: we have

$$\sum x^{i+1} y^j \psi'_{ij} = y^3\varphi'_{02} + yF(\varphi'_2 - \varphi_2) - FF_y\varphi.$$

In particular, using that $F_y = 2y + tx$ and looking at the terms in $y^3\mathbb{k}[E]$, we find that $0 = \varphi'_{02} + (\varphi'_2 - \varphi_2) + 2(\varphi' - \varphi)$, or, rearranging, $\varphi'_{02} = -2\varphi E + \lambda$. ‘‘Integrating’’, we see there exists $\mu \in \mathbb{k}$ such that

$$\varphi_{02} = \varphi(E^2 - E) - \lambda E + \mu.$$

Now, as $FF_y = 2y^3 + 3txy^2 + t^2x^2y$, we must have

$$\sum x^i y^j \psi'_{ij} = ty^2(\varphi'_2 - \varphi_2) - (3ty^2 + t^2xy)\varphi,$$

and, integrating yet another time, we get $\sum x^i y^j \psi_{ij} = \phi(tFE + ty^2(E - E^2)) + \lambda ty^2 E$,

We conclude in this way that every 1-cocycle of degree 1 is cohomologous to one of the form

$$\omega = \varphi\zeta_1 + \lambda\zeta_2 + f\hat{y} + hD\hat{y} + \psi D^2\hat{y} + \mu y^2\hat{x} \quad (20)$$

where ζ_1 and ζ_2 are the cocycles in the statement, $\varphi, \lambda, \psi, \mu \in \mathbb{k}$, $h \in S_1$ and $f \in S_2$.

It is easy to see from the expression we have for δ_1^0 that such a cocycle is a coboundary if and only if it is a scalar multiple of $F\hat{y}$. The upshot of all this is that

$$H^1(S, U)_1 \cong \langle \zeta_1, \zeta_2 \rangle \oplus \mathbb{k}y^2\hat{x} \oplus \left(S_2/(F) \oplus S_1D \oplus \mathbb{k}D^2 \right) \hat{y},$$

as we wanted. \square

5.3. The action of U on $H^\bullet(S, U)$. As we have already computed in Propositions 5.3 and 5.4 the homogeneous components of degrees 0 and 1 of the Hochschild cohomology $H^q(S, U)$ for each q , Proposition 5.2 tells us that in order to compute the second page $E_2^{\bullet, q} = H_S^\bullet(L, H^q(S, U))$ it remains only to find the kernel and the cokernel of $\nabla_D^q : H^q(S, U)_0 \rightarrow H^q(S, U)_1$.

Proposition 5.5. (i) *The kernel of $\nabla_D^0 : H^0(S, U)_0 \rightarrow H^0(S, U)_1$ is \mathbb{k} and its cokernel is S_1 , the subspace of S with basis (x, y) .*

(ii) *The kernel of $\nabla_D^2 : H^2(S, U)_0 \rightarrow H^2(S, U)_1$ is $\mathbb{k}D^2\hat{x} \wedge \hat{y}$ and its cokernel is zero.*

(iii) *The map $\nabla_D^1 : H^1(S, U)_0 \rightarrow H^1(S, U)_1$ is a monomorphism and its cokernel is generated by the classes of the cocycles ζ_1, ζ_6 , and ζ_8 given in Proposition 5.4.*

Proof. Recall that $H^\bullet(S, U)$ is computed from the Koszul resolution P_\bullet of Lemma 4.4, where W is the vector space spanned by x and y . To describe the action of D on $H^\bullet(S, U)$ we need a lifting of $D_S : S \rightarrow S$ to an P_\bullet . We obtain one by letting, for each $q \in \{0, 1, 2\}$, $D_q : P_q \rightarrow P_q$ be the D_S^e -operator such that

$$D_0(1|1) = 0,$$

$$D_1(1|1 \otimes y) = (1|y + y|1 + tx|1) \otimes y + t|y \otimes x, \quad D_1(1|1 \otimes x) = 0,$$

$$D_2(1|1 \otimes x \wedge y) = (1|y + y|1 + tx|1) \otimes x \wedge y,$$

as a straightforward calculation shows. From the description of D_0 we see that the restriction to $S_0 \rightarrow S_1$ of the map $\nabla_D^0 : S \rightarrow S$ is zero, thus proving assertion (i).

We recall from Proposition 5.3 that the homogeneous components of degree 0 and 1 of $H^2(S, U)$ are $D^2\mathbb{k}[E]\hat{x} \wedge \hat{y}$ and $D^3\mathbb{k}[E]\hat{x} \wedge \hat{y}$, respectively. Let us compute the kernel and the cokernel of $\nabla_D^2 : H^2(S, U)_0 \rightarrow H^2(S, U)_1$. We have

$$D_2^\#(D^2\varphi\hat{x} \wedge \hat{y}) = \left([D, D^2\varphi] - D^2\varphi\hat{x} \wedge \hat{y}(D_2(1|1 \otimes x \wedge y)) \right) \hat{x} \wedge \hat{y}$$

and, as in the second term there never appears a higher power of D than D^2 ,

$$D_2^\sharp(D^2\varphi\hat{x} \wedge \hat{y}) \equiv D^3\varphi'\hat{x} \wedge \hat{y} \pmod{\text{Im } \delta_1^1}.$$

The claim in the second item follows from this.

For (iii) we give explicit formulas for the evaluation of $\nabla_D^1 : H^1(S, U)_0 \rightarrow H^1(S, U)_1$ and, at the same time, compute its cokernel. Suppose that ω is a representative of a class in $H^1(S, U)$ chosen as in (20). As $D_1^\sharp(D\hat{y}) = (-F_yD - F)\hat{y}$, we see that up to adding to ω an element in the image of ∇_D^1 we may suppose that $h = h_0x$, for some $h_0 \in \mathbb{k}$.

Let α, β and γ in \mathbb{k} and define $\phi = \alpha y\hat{x} + (\beta x + \gamma y)\hat{y}$. Since $\phi(D_1(1|1 \otimes y))$ is equal to $\gamma xF_x - \alpha yF_x - \beta xF_y$, we have

$$\begin{aligned} D_1^\sharp(\phi) &= ([D, \alpha y] - \phi(D_1(1|x|1)))\hat{x} + ([D, \beta x + \gamma y] - \phi(D_1(1|y|1)))\hat{y} \\ &= \alpha F\hat{x} + (\gamma yF_y + \alpha yF_x + \beta xF_y)\hat{y}. \end{aligned}$$

In view of this, it is easy to see that we may choose α, β and γ in such a way that $\omega + D_1^\sharp(\phi)$, which is a cocycle of the form (20), has $\mu = 0$ and $f = 0$ since $\{yF_x, xF_y, F\}$ spans S_2 .

Let us see that the 1-cocycle ζ_2 belongs to the image of ∇_D^1 . Using the 1-cocycle $\eta_1 = (-yE + D)\hat{x} + tyE\hat{y}$ we get

$$D_1^\sharp(\eta_1)(1|1 \otimes x) = [D, -yE + D] = -FE + yD,$$

and

$$\begin{aligned} D_1^\sharp(\eta_1)(1|1 \otimes y) &= [D, tyE] - \eta_1(D_1(1|1 \otimes y)) \\ &= tFE - tyD - t(-yE + D)y - (tx + y)tyE - tyEy \\ &= -2tyD + ty^2 + t(y^2 + txy), \end{aligned}$$

which belongs to $S_2 + \mathbb{k}yD$. We already know that the elements of $(S_2 + \mathbb{k}yD)\hat{y}$ are coboundaries: it follows that $D_1^\sharp(\eta_1) \equiv (-FE + yD)\hat{x}$ modulo coboundaries. Now, the difference between $D_1^\sharp(\eta_1)$ and ζ_2 is cohomologous to $txyE\hat{x} + ty^2E\hat{y}$, which is in turn equal to $\delta_1^0(-tyE)$. As a consequence of this, we have that $\nabla_D^1(\eta_1)$ is equal to ζ_2 in cohomology.

We conclude from the preceding calculation that $\text{coker}(\nabla_D^1 : H^1(S, U)_0 \rightarrow H^1(S, U)_1)$ is generated by the classes of $\zeta_1, xD\hat{y}$, and $D^2\hat{y}$. Since these classes are linearly independent, the dimension of this cokernel is 3. Finally, we can use the dimension theorem to see that $\nabla_D^1 : H^1(S, U)_0 \rightarrow H^1(S, U)_1$ is a monomorphism. \square

5.4. The second page. We have already made all the computations required for the second page of the spectral sequence.

Proposition 5.6. *The second page of the spectral sequence E_\bullet of Corollary 3.3 converging to the Hochschild cohomology $HH^\bullet(U)$ has dimensions*

$$\dim E_2^{p,q} = \begin{array}{c} \begin{array}{c} q \uparrow \\ 1 \quad 1 \quad 0 \\ 0 \quad 3 \quad 3 \\ 1 \quad 3 \quad 2 \\ \downarrow \\ p \end{array} \end{array} \quad (21)$$

Proof. Let $q \geq 0$ and recall that the q th row $E_2^{\bullet,q}$ is equal to the Lie-Rinehart cohomology $H_S^\bullet(L, H^q(S, U))$. Thanks to Proposition 4.5, $H^q(S, U)$ is an eulerian U -module and we may use Proposition 5.2, which asserts that to obtain $H_S^\bullet(L, H^q(S, U))$ we need only the nullity and rank of $\nabla_D^q : H^q(S, U)_0 \rightarrow H^q(S, U)_1$. This information is provided by Proposition 5.5. \square

Corollary 5.7. *The dimension of $HH^3(U)$ is 3 or 4.*

Proof. The differential in the second page (21) could be non-zero, since neither the domain nor the codomain of the map $d_2^{0,2} : E_2^{0,2} \rightarrow E_2^{2,1}$ are. As $\dim E_2^{0,2} = 1$, the differential $d_2^{0,2}$ is either zero or a monomorphism. If it is zero, the sequence degenerates and using Corollary 3.3 we obtain that $\dim HH^3(U) = 4$; if not, we have $\dim HH^3(U) = 3$. \square

It follows from Corollary 5.7 that to see whether the sequence degenerates or not it is enough to compute the dimension of $HH^3(U)$: this provided in the next proposition.

Proposition 5.8. *The dimension of $HH^3(U)$ is at least 4.*

Proof. The Hochschild cohomology of the algebra of differential operators U on an arrangement of more than five lines is computed in [9] from a complex that we may still use. This complex is given by $U \otimes \Lambda^\bullet V_U^*$, where V_U is the subspace of U spanned by x, y, D and E , or more graphically

$$U \xrightarrow{d^0} U \otimes V_U^* \xrightarrow{d^1} U \otimes \Lambda^2 V_U^* \xrightarrow{d^2} U \otimes \Lambda^3 V_U^* \xrightarrow{d^3} U \otimes \Lambda^4 V_U^*,$$

with differentials such that

$$\begin{aligned} d^2(u \otimes \hat{x} \wedge \hat{y}) &= ([D, u] - \nabla_y^u(F)) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, u] - 2u) \otimes \hat{x} \wedge \hat{y} \wedge \hat{E}; \\ d^2(u \otimes \hat{x} \wedge \hat{E}) &= -[y, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} - [D, u] \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + tuy \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^2(u \otimes \hat{y} \wedge \hat{E}) &= [x, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{E} + ((tx + 2y)u - [y, u] - [D, u]) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\ d^2(u \otimes \hat{x} \wedge \hat{D}) &= -[y, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, u] - 2u) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E}; \\ d^2(u \otimes \hat{y} \wedge \hat{D}) &= [x, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} + ([E, u] - 2u) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \end{aligned}$$

$$\begin{aligned}
d^2(u \otimes \hat{D} \wedge \hat{E}) &= [x, u] \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} + [y, u] \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}; \\
d^3(u \otimes \hat{x} \wedge \hat{y} \wedge \hat{D}) &= (-[E, u] + 3u) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E} \\
d^3(u \otimes \hat{x} \wedge \hat{y} \wedge \hat{E}) &= ([D, u] - (tx + 2y)u + [y, u]) \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}; \\
d^3(u \otimes \hat{x} \wedge \hat{D} \wedge \hat{E}) &= -[y, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}; \\
d^3(u \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}) &= [x, u] \otimes \hat{x} \wedge \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned}$$

As V_U is a homogeneous subspace of U , the grading of U induces on the exterior algebra $\Lambda^\bullet V_U$ an internal grading. There is as well a natural internal grading on the complex $U \otimes \Lambda^\bullet V_U^*$ coming from the grading of U , with respect to which the differentials are homogeneous. Moreover, the inclusion $\mathfrak{X}^\bullet = (U \otimes \Lambda^\bullet V_U^*)_0 \hookrightarrow U \otimes \Lambda^\bullet V_U^*$ of the component of degree zero of the complex $U \otimes \Lambda^\bullet V_U^*$ is a quasi-isomorphism: we will use the complex \mathfrak{X}^\bullet again to compute $HH^3(U)$.

We borrow from our previous calculations the following four cochains in \mathfrak{X}^3 :

$$\begin{aligned}
\omega_1 &= D^2 \hat{x} \wedge \hat{y} \wedge \hat{E} + \left(2D^2 E - 2yDE^2 + F(E^3 - 2E^2 + E)/2 \right) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E} \\
&\quad + \left(-tD^2 E + 2tyDE^2 + tFE^2 \right) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
\omega_2 &= \left(D^2 - 2yDE + y^2(E^2 - E) \right) \otimes \hat{x} \wedge \hat{D} \wedge \hat{E} \\
&\quad + \left(2tyDE + tFE + ty^2(E - E^2) \right) \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
\omega_3 &= D^2 \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}, \\
\omega_4 &= xD \otimes \hat{y} \wedge \hat{D} \wedge \hat{E}.
\end{aligned}$$

It is straightforward to see that these cochains are in fact cocycles. We will now show that the classes of these cocycles are linearly independent, so that $\dim HH^3(U) \geq 4$. We take a linear combination $\omega = \sum_{i=1}^4 \lambda_i \omega_i$ with $\lambda_1, \dots, \lambda_4 \in \mathbb{k}$ and suppose that there exists a cochain ξ in \mathfrak{X}^2 such that $d^2(\xi) = \omega$. Since the component of ω in $\hat{x} \wedge \hat{y} \wedge \hat{D}$ is zero, we may write

$$\xi = u \otimes \hat{x} \wedge \hat{E} + v \otimes \hat{y} \wedge \hat{E} + w \otimes \hat{D} \wedge \hat{E},$$

with u, v and w in U_1 , and there exist then $\alpha_i, \beta_i, \gamma_i \in \mathbb{k}[E]$ with $1 \leq i \leq 3$ such that

$$u = x\alpha_1 + y\alpha_2 + D\alpha_3, \quad v = x\beta_1 + y\beta_2 + D\beta_3, \quad w = x\gamma_1 + y\gamma_2 + D\gamma_3.$$

We now examine each component of the equality $d^2(\xi) = \omega$. In $\hat{x} \wedge \hat{y} \wedge \hat{D}$ there is nothing to see. In $\hat{x} \wedge \hat{y} \wedge \hat{E}$ we have $-[y, u] + [x, v] = \lambda_1 D^2$, or

$$-xy\alpha'_1 - y^2\alpha'_2 - yD\alpha'_3 - F(\alpha'_3 - \alpha_3) + x^2\beta'_1 + xy\beta'_2 + xD\beta'_3 = \lambda_1 D^2.$$

This is an equality in U_2 , which we may decompose as $\bigoplus_{i+j+k=2} x^i y^j D^k \mathbb{k}[E]$. Looking at $D^2 \mathbb{k}[E]$ we get $\lambda_1 = 0$, from $yD\mathbb{k}[E]$, $x^2 \mathbb{k}[E]$ and $xD\mathbb{k}[E]$ we obtain $\alpha'_3 = \beta'_1 = \beta'_3 = 0$ and $xy\mathbb{k}[E]$ and $y^2 \mathbb{k}[E]$ tell us that $\alpha_3 = \alpha'_2$ and $\beta'_2 = \alpha'_1 - t\alpha_3$.

In $\hat{x} \wedge \hat{D} \wedge \hat{E}$, equation $d^2(\xi) = \omega$ reads

$$-xD\alpha'_1 - F\alpha_2 - yD\alpha'_2 + x^2\gamma'_1 + xy\gamma'_2 + xD\gamma'_3 = \lambda_2(D^2 - 2yDE + y^2(E^2 - E)).$$

The component in $D^2\mathbb{k}[E]$ of this equality is $0 = \lambda_2$. From $xD\mathbb{k}[E]$ and $yD\mathbb{k}[E]$ we obtain $\gamma'_3 = \alpha'_1$ and $\alpha'_2 = 0$ and from $x^2\mathbb{k}[E]$, $xy\mathbb{k}[E]$ and $y^2\mathbb{k}[E]$ we get $\gamma'_1 = 0$, $\gamma'_2 = t\alpha_2$ and $\alpha_2 = 0$. In particular, that $\alpha_2 = 0$ implies that $\alpha_3 = 0$ and that

$$\beta'_2 = \alpha'_1 = \gamma'_3 \tag{22}$$

We finally look at the component in $\hat{y} \wedge \hat{D} \wedge \hat{E}$ of $d^2(\xi) = \omega$, which is

$$tuy + (tx + 2y)v - [y, v] - [D, v] + [y, w] = \lambda_3 D^2 + \lambda_4 xD.$$

This is an equality in $U_2 = \bigoplus_{i+j+k=2} x^i y^j D^k \mathbb{k}[E]$. In $D^2\mathbb{k}[E]$ we have $0 = \lambda_3$, and in $yD\mathbb{k}[E]$

$$2\beta_3 yD - yD\beta'_2 + yD\gamma'_3 = 0,$$

which in the light of (22) implies $\beta_3 = 0$. With this at hand we see that in $xD\mathbb{k}[E]$ it only remains $0 = \lambda_4$.

We have seen at this point that the only coboundary among the cocycles of the form $\omega = \sum_{i=1}^4 \lambda_i \omega_i$ is $\omega = 0$. This shows that the classes of $\omega_1, \dots, \omega_4$ are linearly independent, thus finishing the proof. \square

Corollary 5.9. *Let \mathcal{A} be a central arrangement of three lines. The Hilbert series of $HH^\bullet(\text{Diff}\mathcal{A})$ is*

$$h_{HH^\bullet(\text{Diff}\mathcal{A})}(t) = 1 + 3t + 6t^2 + 4t^3.$$

Proof. Proposition 5.8 implies at once that the spectral sequence degenerates at E_2 . The dimensions in the statement are a consequence of the convergence of the sequence in Corollary 3.3 and the information in Proposition 5.6. \square

As a consequence of the information we have gathered so far we can easily describe the Lie algebra structure on $HH^1(\text{Diff}\mathcal{A})$. Let us, again, call $U = \text{Diff}\mathcal{A}$ and recall that $HH^1(U)$ is isomorphic to the space $\text{OutDer}U$ of outer derivations of U , that is, the quotient of the derivations of U modulo inner derivations, and that the commutator of derivations induces a Lie algebra structure on $\text{OutDer}U$. We know from [9, Proposition 4.2] that if $f \in S_1$ divides x^F then there is a derivation $\partial_f : U \rightarrow U$ such that $\partial_f(x) = \partial_f(y) = 0$, $\partial_f(D) = \frac{F}{f} \partial_y f$ and $\partial_f(E) = 1$. Let then $f_1 = x$, $f_2 = y$ and $f_3 = tx + y$ and put $\partial_i := \partial_{f_i}$ for $1 \leq i \leq 3$.

Corollary 5.10. *Let \mathcal{A} be a central arrangement of three lines. The Lie algebra of outer derivations of $\text{Diff}\mathcal{A}$ together with the commutator is an abelian Lie algebra of dimension three generated by the classes of the derivations ∂_1 , ∂_2 and ∂_3 .*

Proof. We claim that the classes of ∂_1 , ∂_2 and ∂_3 are linearly independent in $\text{OutDer}(U)$. Indeed, let $u \in U$ and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{k}$ be such that

$$\sum \lambda_i \partial_i(v) = [u, v] \quad \text{for every } v \in U. \quad (23)$$

Evaluating (23) on each $s \in S$ the left side vanishes and therefore Proposition 5.3 tells us that $u \in S$. Write $u = \sum_{j \geq 0} u_j$ with $u_j \in S_j$. Evaluating now (23) on E we obtain $\sum_i \lambda_i = -\sum_j j u_j$. In each homogeneous component S_j with $j \neq 0$ we have $j u_j = 0$ and therefore $u \in S_0 = \mathbb{k}$ and $\sum_i \lambda_i = 0$. This equation and the one we get evaluating (23) on D , that is, $\lambda_2(tx + y) + \lambda_3 y = 0 \in S_1$, finally tell us that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

The classes of ∂_1 , ∂_2 and ∂_3 span $\text{OutDer } U$ because, thanks to Corollary 5.9, its dimension is three. The composition $\partial_i \circ \partial_j : U \rightarrow U$ is evidently equal to zero for any $1 \leq i, j \leq 3$, as a straightforward calculation shows, and therefore the Lie algebra structure in $\text{OutDer } U$ vanishes. \square

It is possible also to use the spectral sequence of Corollary 3.3 to obtain $HH^\bullet(\text{Diff } \mathcal{A})$ for arrangements with any $l \geq 3$, but we will not perform this calculation here. The result is

$$h_{HH^\bullet(\text{Diff } \mathcal{A})}(t) = \begin{cases} 1 + lt + 2lt^2 + (l+1)t^3, & \text{if } l = 3, 4; \\ 1 + lt + (2l-1)t^2 + lt^3, & \text{if } l \geq 5. \end{cases}$$

This shows that the case in which l is 3 or 4 is *genuinely* different to that in which $l \geq 5$. If $l \leq 2$, the algebra $\text{Diff } \mathcal{A}$ is not very interesting, since it is isomorphic to algebras with well-known Hochschild cohomology —see [9, §3.8].

6. OTHER APPLICATIONS

6.1. The Hochschild cohomology of a family of subalgebras of the Weyl algebra. Let \mathbb{k} be a field of characteristic zero, fix a nonzero $h \in \mathbb{k}[x]$ and consider the algebra A_h with presentation

$$\frac{\mathbb{k}\langle x, y \rangle}{(yx - xy - h)}.$$

Setting $h = 1$ the algebra A_h is the Weyl algebra A_1 that already appeared in Example 1.4, when $h = x$ it is the universal enveloping algebra of the two-dimensional non-abelian Lie algebra and if $h = x^2$, it is the *Jordan plane* studied in [1].

We let $S = \mathbb{k}[x]$ and consider the Lie algebra L freely generated by $y = h \frac{d}{dx}$ as an S -submodule of $\text{Der } S$. It is straightforward to see that (S, L) is a Lie–Rinehart algebra whose enveloping algebra U is isomorphic to A_h . We will use the spectral sequence of Corollary 3.3 to compute the Hochschild cohomology $HH^\bullet(A_h)$ of A_h : we will describe explicitly the second page and find that the spectral sequence degenerates at that page.

6.1.1. *The Hochschild cohomology $H^\bullet(S, U)$.* The augmented Koszul complex

$$P_\bullet : 0 \longrightarrow S^e \xrightarrow{\delta_1} S^e \xrightarrow{\varepsilon} S$$

with $\delta_1(s \otimes t) = sx \otimes t - s \otimes xt$ and augmentation $\varepsilon(s \otimes t) = st$ is an S^e -projective resolution of S and therefore the Hochschild cohomology $H^\bullet(S, U)$ is, after identifying $\text{Hom}_{S^e}(S^e, U)$ with U , the cohomology of the complex $U \xrightarrow{\delta} U$ with differential $\delta(u) = [x, u]$.

Proposition 6.1. *There are isomorphisms of vector spaces*

$$H^0(S, U) \cong S, \quad H^1(S, U) \cong U/hU, \quad H^q(S, U) = 0 \quad \text{if } q \geq 2.$$

Proof. The isomorphisms in the statement come, of course, of the computation of the cohomology of $U \xrightarrow{\delta} U$. Let us first deal with $\ker \delta$. Writing $u = \sum_{i=0}^r f_i y^i$ with $f_1, \dots, f_r \in S$ and r the greatest index such that $f_r \neq 0$, we have that

$$\delta(u) = r f_r h y^{r-1} + v_u \tag{24}$$

for some $v_u \in \bigoplus_{i=0}^{r-2} S y^i$. If $\delta(u) = 0$ then its principal symbol $r f_r h y^{r-1}$ must be equal to zero and then, because the field has characteristic zero, either $r = 0$ or $f_r = 0$. This second possibility contradicts our assumptions, and therefore $r = 0$ and $u \in S$. That, reversely, S is contained in the kernel of δ is evident.

The second claim of the statement follows from the fact that the image of δ is the right ideal generated by h , that is, hU . For this, we can see that $hS y^i$ belongs to the image of δ for every $i \geq 0$ with a straightforward inductive argument using (24). \square

6.1.2. *The action of U on $H^\bullet(S, U)$.* As S acts just by left multiplication, to determine the action of U it is enough to explicit that of y : for this have at hand Remark 2.5.

Proposition 6.2. *Under the isomorphisms $H^0(S, U) \cong U$ and $H^1(S, U) \cong U/hU$ of Proposition 6.1, the action of L on $H^\bullet(S, U)$ is determined by*

$$\begin{aligned} \nabla_y^0(s) &= h s' & \text{for } s \in S; \\ \nabla_y^1(\bar{u}) &= -\overline{h' u} & \text{for } u \in U, \end{aligned}$$

where the overline denotes class modulo hU .

Proof. We use Example 2.7 to see that y acts on $H^0(S, U) = S$ as in the statement. To describe its action on $H^1(S, U)$ we need a lifting $y_\bullet = (y_0, y_1)$ of $y_S : S \rightarrow S$ to P_\bullet . Let us define

$$y_0(s \otimes t) = h s' \otimes 1 + 1 \otimes h t, \quad y_1(s \otimes t) = h s' \otimes 1 + 1 \otimes h t' + s \Delta(h) t,$$

where $\Delta : S \rightarrow S^e$ is the unique derivation of S such that $\Delta(x) = 1 \otimes 1$, that is, it is the only linear map such that $\delta(x^j) = \sum_{s+t=j+1} x^s \otimes x^t$ if $j \geq 0$. We readily see that y_0 and

y_1 are y_S^e -operators and that the diagram

$$\begin{array}{ccccc} S^e & \xrightarrow{\delta_1} & S^e & \xrightarrow{\varepsilon} & S \\ y_1 \uparrow & & y_0 \uparrow & & y \uparrow \\ S^e & \xrightarrow{\delta_1} & S^e & \xrightarrow{\varepsilon} & S \end{array}$$

commutes, and thus the pair (y_0, y_1) is in fact one of the y_S^e -liftings we were looking for.

We now compute $y_1^\sharp : \text{Hom}_{S^e}(S^e, U) \rightarrow \text{Hom}_{S^e}(S^e, U)$ following (8), and for that we let $\phi \in \text{Hom}_{S^e}(S^e, U)$. Bearing in mind the isomorphism $\text{Hom}_{S^e}(S^e, U) \cong U$ induced by the evaluation in $1 \otimes 1$, we need only compute

$$y_1^\sharp(\phi)(1 \otimes 1) = [y, \phi(1 \otimes 1)] - \phi(\Delta(h)).$$

Assuming without losing generality that $\phi(1 \otimes 1) = fy^i$ and $h = x^j$ for $f \in S$ and $i, j \geq 0$, we obtain

$$\begin{aligned} y_1^\sharp(\phi)(1 \otimes 1) &= [y, fy^i] - \sum_{s+t=j+1} x^s f y^i x^t = h f' y^i - \sum_{s+t=j+1} x^s f (x^t y^i + [y^i, x^t]) \\ &\equiv -j x^{j-1} f \pmod{hU}, \end{aligned}$$

since $[y^i, x^t] \in hU$ for all $i, t \geq 0$. Taking class in cohomology and identifying ϕ with $\phi(1 \otimes 1)$, we get

$$\nabla_y^1(\overline{fy^i}) = -\overline{h'fy^i},$$

and the stated result follows from this. \square

6.1.3. The Lie-Rinehart cohomology. Let us now compute $H_S^\bullet(L, H^i(S, U))$ for each $i \in \mathbb{Z}$. Using the complex in Proposition 1.6 to compute Lie-Rinehart cohomology of S , we see that this is the cohomology of the complex

$$H^i(S, U) \xrightarrow{\nabla_y^i} H^i(S, U).$$

Proposition 6.3. *Let $d = \gcd(h, h')$ and let I be the ideal of $S/(h)$ generated by the class of h/d . There are isomorphisms of vector spaces*

$$\begin{aligned} H_S^0(L, H^0(S, U)) &\cong \mathbb{k}, & H_S^1(L, H^0(S, U)) &\cong S/(h), \\ H_S^0(L, H^1(S, U)) &\cong I[y], & H_S^1(L, H^1(S, U)) &\cong S/(d)[y]. \end{aligned}$$

and $H_S^p(L, H^q(S, U)) = 0$ if $p, q \geq 2$.

Proof. We make use of the explicit description of ∇_y^i in Proposition 6.2. For $i = 0$, this amounts to the cohomology of $S \xrightarrow{y} S$, and we readily see that the kernel of this map is \mathbb{k} and its image, hS .

Consider now the case in which $i = 1$ and recall that $H^1(S, U)$ is isomorphic to U/hU . As U/hU is the quotient of the free noncommutative algebra in x and y by the relations $xy - yx = h$ and $h = 0$, we may identify $H^1(S, U)$ with $\frac{S}{(h)}[y]$.

For each $f \in S$ we write \tilde{f} its class in $S/(h)$. This way, given $v \in \frac{S}{(h)}[y]$, there are $f_0, \dots, f_r \in \mathbb{k}[x]$ such that $v = \sum_{i=0}^r \tilde{f}_i y^i$ and, as our findings on ∇_y^1 of Proposition 6.2 allow us to see,

$$\nabla_y^1(v) = - \sum_{i=0}^r \widetilde{h' f_i} y^i.$$

It is immediate that the cokernel of ∇_y^1 is $\frac{S}{(h, h')}[y]$. To compute its kernel, let us suppose that $\nabla_y^1(u) = 0$. For each $i \in \{0, \dots, r\}$ we have that h divides $h' f_i$ and therefore, if d denotes the greatest common divisor of h and h' , we have that h/d divides f_i . Denoting by I the ideal of $\frac{S}{(h)}$ generated by the class of h/d , we conclude that $H_S^0(L, H^1(S, U))$ is isomorphic to the space of polynomials $I[y]$ with coefficients in I . \square

The conclusion of this calculation is the following description of the Hochschild cohomology of U —this time we write A_h instead of U .

Proposition 6.4. *There are isomorphisms of vector spaces*

$$HH^i(A_h) \cong \begin{cases} \mathbb{k} & \text{if } i = 0; \\ S/(h) \oplus I[y] & \text{if } i = 1; \\ \frac{S}{(d)}[y] & \text{if } i = 2; \\ 0 & \text{otherwise,} \end{cases}$$

where d stands for the greatest common divisor of h and its derivative h' and I is the ideal of $\frac{S}{(h)}$ generated by the class of h/d .

Proof. The spectral sequence E_\bullet of Corollary 3.3 converges to $HH^\bullet(A_h)$ and its second page, given by $E_2^{p,q} = H_S^p(L, H^q(S, U))$, is completely computed in Proposition 6.3. The sequence degenerates because $E_2^{p,q} = 0$ if $p, q \geq 2$. \square

The first cohomology space had already been obtained, in other words, in [3, Theorem 5.7.(iii)] and the second one in [11, Corollary 3.11]. However, the spectral sequence argument provides conceptual simplifications and, as a consequence of that, this computation is significantly shorter.

6.2. The Van den Bergh duality property for U . An appropriate specialization of Corollary 3.3 allows us to recover one of the main results of [10], which we recall after the following preliminary definition. Let $n \geq 0$. An algebra A has *Van den Bergh duality* of dimension n if A has a resolution of finite length by finitely generated projective A -bimodules and there exists an invertible A -bimodule D such that there is an isomorphism of A -bimodules

$$\text{Ext}_{A^e}^i(A, A \otimes A) = \begin{cases} 0 & \text{if } i \neq n; \\ D & \text{if } i = n. \end{cases}$$

The Van den Bergh duality property for an algebra A is important because, as can be seen in [21], it relates the Hochschild cohomology of A with its homology in a way analogue to Poincaré duality: indeed, for each A -bimodule M it produces a canonical isomorphism $H^i(A, M) \rightarrow H_{n-i}(A, D \otimes_A M)$.

Let us consider the left U -module structure on $\Lambda_S^d L^\vee \otimes_S D$ discussed in the first section of [10]. If F is the functor from left U -modules to U^e -modules in (13) then evidently $F(\Lambda_S^d L^\vee \otimes_S D)$ becomes an U^e -module.

Theorem 6.5. *Let (S, L) be a Lie–Rinehart algebra such that S has Van den Bergh duality in dimension n and L is finitely generated and projective with constant rank k as an S -module and let $L^\vee = \text{Hom}_S(L, S)$. The enveloping algebra U of the algebra has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of U^e -modules*

$$\text{Ext}_{U^e}^{n+d}(U, U^e) \cong F(\Lambda_S^d L^\vee \otimes_S D).$$

Lemma 6.6. *Let A be an algebra and T and P two A -modules such that T admits a projective resolution by finitely generated A -modules and P is flat. There is an isomorphism*

$$\text{Ext}_A^\bullet(T, P) \cong \text{Ext}_A^\bullet(T, A) \otimes_A P.$$

Proof of Lemma 6.6. Let Q_\bullet be such a resolution of T . For each $i \geq 0$, the evident map from $\text{Hom}_A(Q_i, A) \otimes_A P$ to $\text{Hom}_A(Q_i, P)$ is an isomorphism because Q_i is finitely generated and projective. As P is flat, the cohomology of the complex $\text{Hom}_A(Q_\bullet, A) \otimes_A P$ is isomorphic to $\text{Ext}_A^\bullet(T, A) \otimes_A P$. \square

Proof of Theorem 6.5. The homological smoothness of U follows from Lemma 5.1.2 of [10], whose proof does not depend on this theorem.

Let us write D for the dualizing bimodule $\text{Ext}_{S^e}^n(S, S^e)$. We take, specializing Corollary 3.3, $M = U^e$ to obtain a spectral sequence E_\bullet such that

$$E_2^{p,q} = H_S^p(L, H^q(S, U^e)) \implies H^{p+q}(U, U^e).$$

Let us first deal with $H^q(S, U^e)$. As we observed in the proof of Proposition 2.1, the U^e -module U^e is S^e -projective and, since S has Van den Bergh duality, it admits a resolution by finitely generated projective S^e -modules. We may therefore use Lemma 6.6 to see that

$$H^q(S, U^e) \cong H^q(S, S^e) \otimes_{S^e} U^e,$$

which is zero if $q \neq n$ and isomorphic to $D \otimes_{S^e} U^e$ if $q = n$. As a consequence of this, our spectral sequence E_\bullet degenerates at its second page and thus $H^{p+n}(U, U^e)$ is isomorphic to $H_S^p(L, D \otimes_{S^e} U^e)$ for each $p \in \mathbb{Z}$.

The Chevalley–Eilenberg complex from Proposition 1.6 is an U -projective resolution of S by finitely generated modules. On the other hand, the dualizing module D is S -projective because it is invertible —see Chapter 6 in the book [4] by F. Anderson and

K. Fuller. We conclude that the U -module $D \otimes_{S^e} U^e$ is projective, and we can apply Lemma 6.6 we obtain an isomorphism

$$H_S^\bullet(L, D \otimes_{S^e} U^e) \cong H_S^\bullet(L, U) \otimes_U (D \otimes_{S^e} U^e).$$

Now, the hypotheses on L are such that Theorem 2.10 in [7] tells us that $H_S^p(L, U)$ is zero if $p \neq d$ and is isomorphic to $\Lambda_S^d L^\vee$ if $p = d$, so that

$$H^i(U, U^e) \cong \begin{cases} \Lambda_S^d L^\vee \otimes_U (D \otimes_{S^e} U^e), & \text{if } i = n + d; \\ 0 & \text{otherwise.} \end{cases}$$

The dualizing bimodule of U is therefore isomorphic to $\Lambda_S^d L^\vee \otimes_U (D \otimes_{S^e} U^e)$, or, as an immediate application of Lemma 3.5.2 in [10] shows, to $F(\Lambda_S^d L^\vee \otimes_S D)$. \square

REFERENCES

- [1] Nicolás Andruskiewitsch, Iván Angiono, and István Heckenberger, *Liftings of Jordan and super Jordan planes*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 3, 661–672, DOI 10.1017/s0013091517000402. MR3834727
- [2] Georgia Benkart, Samuel A. Lopes, and Matthew Ondrus, *A parametric family of subalgebras of the Weyl algebra I. Structure and automorphisms*, Trans. Amer. Math. Soc. **367** (2015), no. 3, 1993–2021, DOI 10.1090/S0002-9947-2014-06144-8.
- [3] Georgia Benkart, Samuel A. Lopes, and Matthew Ondrus, *Derivations of a parametric family of subalgebras of the Weyl algebra*, J. Algebra **424** (2015), 46–97, DOI 10.1016/j.jalgebra.2014.11.007.
- [4] Frank W. Anderson and Kent R. Fuller, *Rings and categories of modules*, 2nd ed., Graduate Texts in Mathematics, vol. 13, Springer-Verlag, New York, 1992. MR1245487
- [5] Francisco J. Calderón-Moreno, *Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor*, Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 5, 701–714, DOI 10.1016/S0012-9593(01)80004-5.
- [6] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.
- [7] Johannes Huebschmann, *Duality for Lie–Rinehart algebras and the modular class*, J. Reine Angew. Math. **510** (1999), 103–159, DOI 10.1515/crll.1999.043. MR1696093
- [8] Johannes Huebschmann, *Poisson cohomology and quantization*, J. Reine Angew. Math. **408** (1990), 57–113, DOI 10.1515/crll.1990.408.57.
- [9] Francisco Kordon and Mariano Suárez-Álvarez, *Hochschild cohomology of algebras of differential operators tangent to a central arrangement of lines* (2018), available at [arXiv:1807.10372](https://arxiv.org/abs/1807.10372). Accepted for publication by Documenta Mathematica.
- [10] Thierry Lambre and Patrick Le Meur, *Duality for differential operators of Lie–Rinehart algebras*, Pacific J. Math. **297** (2018), no. 2, 405–454, DOI 10.2140/pjm.2018.297.405. MR3893434
- [11] Samuel A. Lopes and Andrea Solotar, *Lie structure on the Hochschild cohomology of a family of subalgebras of the Weyl algebra* (2019), available at [arXiv:1903.01226](https://arxiv.org/abs/1903.01226).
- [12] L. Narváez Macarro, *Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors*, Singularities I, 2008, pp. 245–269, DOI 10.1090/conm/474/09259.
- [13] Jet Nestruev, *Smooth manifolds and observables*, Graduate Texts in Mathematics, vol. 220, Springer-Verlag, New York, 2003.
- [14] Peter Orlik and Louis Solomon, *Combinatorics and topology of complements of hyperplanes*, Invent. Math. **56** (1980), no. 2, 167–189, DOI 10.1007/BF01392549.

- [15] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften, vol. 300, Springer-Verlag, Berlin, 1992.
- [16] George S. Rinehart, *Differential Forms on General Commutative Algebras*, Transactions of the American Mathematical Society **108** (1963), no. 2, 195–222.
- [17] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR586450
- [18] Hiroaki Terao, *Free arrangements of hyperplanes and unitary reflection groups*, Proc. Japan Acad. Ser. A Math. Sci. **56** (1980), no. 8, 389–392. MR596011
- [19] Mariano Suárez-Álvarez, *The algebra of differential operators tangent to a hyperplane arrangement* (2018), available at [arXiv:1806.05410](https://arxiv.org/abs/1806.05410).
- [20] Mariano Suárez-Álvarez, *A little bit of extra functoriality for Ext and the computation of the Gerstenhaber bracket*, J. Pure Appl. Algebra **221** (2017), no. 8, 1981–1998, DOI 10.1016/j.jpaa.2016.10.015. MR3623179
- [21] Michel Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348, DOI 10.1090/S0002-9939-98-04210-5. Erratum, *ibid* **130** (2002), no. 9, 2809–2810; MR1900889. MR1443171
- [22] C. Weibel, *An introduction to Homological algebra*, Cambridge University Press, 1994.
- [23] Jonathan Wiens and Sergey Yuzvinsky, *De Rham cohomology of logarithmic forms on arrangements of hyperplanes*, Trans. Amer. Math. Soc. **349** (1997), no. 4, 1653–1662, DOI 10.1090/S0002-9947-97-01894-1.

LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL, UMR6620 CNRS, UNIVERSITÉ CLERMONT AUVERGNE, CAMPUS DES CÉZEAUX, 3 PLACE VASARELY, 63178 AUBIÈRE CEDEX, FRANCE

E-mail address, F. Kordon: francisco.kordon@uca.fr

E-mail address, Th. Lambre: thierry.lambre@uca.fr