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Abstract. In this paper we establish some explicit and sharp estimates of the spectral gap and the log-Sobolev constant for mean field particles system, uniform in the number of particles, when the confinement potential have many local minimums. Our uniform log-Sobolev inequality, based on Zegarlinski's theorem for Gibbs measures, allows us to obtain the exponential convergence in entropy of the McKean-Vlasov equation with an explicit rate constant, generalizing the result of [10] by means of the displacement convexity approach, or [19, 20] by Bakry-Emery technique or the recent [9] by dissipation of the Wasserstein distance.

Key words : Poincaré inequality, logarithmic Sobolev inequality, mean field particle models, McKean-Vlasov equation.

MSC 2010 :

1. Introduction

Functional inequalities such as Poincaré or logarithmic Sobolev inequalities have nowadays an important impact on various fields of mathematics (probability, PDE, statistics,...) due to their various properties such as convergence to equilibrium (in L^2 or in entropy) or concentration of measure (exponential or gaussian). We refer to the beautiful book [3] for an introduction (and more) to the subject as well as bibliographical references. Let us introduce these two inequalities. Let μ be a probability measure on $\mathbb{R}^d$, we say that the probability measure μ satisfies a Poincaré (or equivalently spectral gap) inequality with (optimal) constant λ_μ if for all smooth functions f we have

$$(PI) \qquad \lambda_1(\mu)\,\mathrm{Var}_\mu(f) \le \int |\nabla f|^2 d\mu, \tag{1}$$

where $\mathrm{Var}_\mu(f) := \int f^2 d\mu - \left(\int f d\mu\right)^2$ denotes the variance of f wrt μ and a logarithmic Sobolev inequality with (optimal) constant ρ_μ if for all smooth functions f we have

$$(LSI) \qquad \rho_{LS}(\mu)\,\mathrm{Ent}_\mu(f^2) \le 2\int |\nabla f|^2 d\mu, \tag{2}$$

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where $\mathrm{Ent}_\mu(f^2) := \int f^2 \log(f^2/\int f^2 d\mu)d\mu$ denotes the entropy of f^2 with respect to (w.r.t. in short) μ. A famous condition to verify those inequalities is the Bakry-Emery Γ_2 criterion which says that if $d\mu = e^{-V}dx$ on $\mathbb{R}^n$, Hess $V \geq \kappa Id > 0$, then $\lambda_1(\mu) \geq \rho_{LS}(\mu) \geq \kappa$.
One crucial property of these two inequalities is the tensorization (or dimension free), i.e. if μ satisfies a Poincaré or a logarithmic Sobolev inequality then $\mu^{\otimes N}$ satisfies the same inequality with the same constant (and thus independent of N) leading for example to (non asymptotic) gaussian deviation inequalities refining central limit inequalities or convergence to equilibrium independent of the number of particles. However interesting physical systems are far from being independent, so that there exists a huge literature devoted to the obtention of functional inequalities such as Poincaré or logarithmic Sobolev inequalities, in particular to assess convergence to equilibrium, in various dependent settings such as (discrete or continuous) spin systems [22, 23, 24, 31, 32, 7, 8, 29, 30, 18, 4] (see also [17] for a survey) or mean field models [19, 20, 11, 14, 15] with a particular emphasis on the dependence on the number of spins or particles.
We will focus our attention on mean field particles system. To this end, consider the $N(\geq 2)$ interacting particles system of mean field type :

$$dX_i^N(t) = \sqrt{2}dB_i(t) - \nabla V(X_i^N(t))dt - \frac{1}{N-1}\sum_{j\neq i}\nabla_x W(X_i^N(t), X_j^N(t))dt, \; i = 1, \cdots, N \quad (3)$$

where $B_1(t), \cdots, B_N(t)$ are N independent Brownian motions taking values in $\mathbb{R}^d$, the confinement potential V is a function on $\mathbb{R}^d$ of class C^2, and the interaction potential W is a function on $\mathbb{R}^d \times \mathbb{R}^d$ of class C^2. Its generator $\mathcal{L}^{(N)}$ is given by

$$\begin{aligned}\mathcal{L}^{(N)} f(x_1, \cdots, x_N) &= \sum_{i=1}^N \mathcal{L}_i^{(N)} f(x_1, \cdots, x_N)\\ \mathcal{L}_i^{(N)} f(x_1, \cdots, x_N) &:= \Delta_i f(x_1, \cdots, x_N) - \nabla_i V(x_i)\cdot\nabla_i f(x_1, \cdots, x_N)\\ &\qquad - \frac{1}{N-1}\sum_{j\neq i}(\nabla_x W)(x_i, x_j)\cdot\nabla_i f(x_1, \cdots, x_N)\end{aligned} \quad (4)$$

for any smooth function f on $(\mathbb{R}^d)^N$, where ∇_i denotes the gradient w.r.t. x_i, Δ_i the Laplacian w.r.t. x_i, and $x \cdot y = \langle x, y\rangle$ denotes the Euclidean inner product.
The unique invariant probability measure of (3) is

$$\mu^{(N)}(dx_1, \cdots, dx_N) = \frac{1}{Z_N}\exp\{-H_N(dx_1, \cdots, dx_N)\}\, dx_1 \cdots dx_N \quad (5)$$

where

$$H_N(x_1, \cdots, x_N) := \sum_{i=1}^N V(x_i) + \frac{1}{N-1}\sum_{1\leq i<j\leq N} W(x_i, x_j)$$

is the Hamiltonian, and Z_N is the normalization constant called *partition function* in statistical mechanics, which is assumed to be finite throughout the paper. Without interaction (i.e. $W = 0$ or constant), $\mu^{(N)} = \alpha^{\otimes N}$ (i.e. the particles are independent), where

$$d\alpha(x) = \frac{1}{C}e^{-V(x)}dx, \; C = \int e^{-V(x)}dx.$$

Our first major goal is to get uniform (in the number of particles N) Poincaré or logarithmic Sobolev inequalities for the measure $\mu^{(N)}$ under tractable conditions. Malrieu [19] used Bakry-Emery's Γ_2 technique to establish a logarithmic Sobolev inequality for the mean field case thus requiring uniform convexity assumption for V and W. Recent techniques such as Lyapunov conditions (see [2, 1] for example) are usually inefficient to get dimension-free results. For each of these inequalities we require a uniform bound for the spectral gap or the logarithmic Sobolev constants of the one particle conditional distribution. To bypass the perturbation techniques, our main assumptions for Poincaré inequality will be of two sorts: for the confinement potential we will need some linear growth at infinity as well as a lipschitzian spectral gap property (see Section 2 for details) which will be sufficient to get a Poincaré inequality for the one particle conditional distribution, and for the interaction potential a lower bound on the "extra diagonal" Hessian of W, leading to new and sharp results. A particular emphasis will be made on Curie-Weiss model and on interaction potential of the form $W(x,y) = W_0(x-y)$. The proof will repose on some refinement of the ideas of Ledoux [18]. For the logarithmic Sobolev inequality we will consider a translation of Zegarlinski's condition (see [31]) for mean field model which relies on the smallness of the product of the Lipschitzian spectral gap and of the infinite norm of the Hessian of the interaction potential. One of our interest to consider logarithmic Sobolev inequality for mean field particles system is to get an exponential entropic decay for the limit non-linear McKean-Vlasov equation. Indeed, consider the non-linear McKean-Vlasov equation with an internal potential $V : \mathbb{R}^d \to \mathbb{R}$ and an interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (between two particles) so that $W(x,y) = W(y,x)$:

$$\partial_t \nu_t = \Delta \nu_t + \nabla \cdot (\nu_t \nabla V) + \nabla \cdot (\nu_t \nabla (W \circledast \nu_t)) \tag{6}$$

where $(\nu_t)_{t\geq 0}$ is a flow of probability measures on $\mathbb{R}^d$ with ν_0 given, ∇ is the gradient, $\nabla\cdot$ is the divergence, and

$$(W \circledast \nu)(x) = \int_{\mathbb{R}^d} W(x,y) d\nu(y). \tag{7}$$

It corresponds to the self-interacting diffusion

$$dX_t = \sqrt{2} dB_t - \nabla V(X_t)dt - \nabla W \circledast \nu_t(X_t)dt \tag{8}$$

where ν_t is the law of X_t. It can be seen through the propagation of chaos phenomenon (see [25] for example) that the law of $X_1^N(t)$ converges to the one of X_t as the number of particles N tends to infinity (for each $t > 0$). Via the logarithmic Sobolev inequality for the mean field particles system and a quite technical passage to the limit, we will be able to prove entropic convergence to equilibrium for the non-linear McKean-Vlasov SDE generalizing results of [10, 9].

Let us finish this introduction by the plan of the paper. In the next section, we will present our set of assumptions and the main results of the paper concerning uniform Poincaré or logarithmic Sobolev inequality of mean field particles system as well as exponential convergence to equilibrium for McKean-Vlasov SDE (8). Section 3 presents the Lipschitzian spectral gap for conditional distribution needed in the proof of the uniform Poincaré inequality detailed in Section 4. The translation of Zegarlinski's condition and thus the proof of uniform logarithmic Sobolev inequality are the core of Section 5. The exponential convergence to equilibrium of McKean-Vlasov SDE is finally detailed in the last Section 6.

2. Main results

2.1. **Framework and main assumptions.** Throughout the paper we work in the following framework.

(H1) The confinement potential $V : \mathbb{R}^d \to \mathbb{R}$ is C^2-smooth, its Hessian $\mathrm{Hess}(V) = \nabla^2 V = (\partial_{x_k}\partial_{x_l} V)_{1\le k,l\le d}$ of V is bounded from below and there are two positive constants c_1, c_2 such that

$$x \cdot \nabla V(x) \geq c_1|x|^2 - c_2,\ x \in \mathbb{R}^d. \tag{9}$$

(H2) The pairwise interaction potential $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is C^2-smooth such that its Hessian $\nabla^2 W$ is bounded and

$$\iint \exp\left(-[V(x) + V(y) + \lambda W(x,y)]\right) dxdy < +\infty,\ \forall \lambda > 0.$$

(H3) **(Lipschitzian spectral gap condition for one particle)** the following Lipschitzian constant (for the marginal conditional distribution of one particle) is finite

$$c_{Lip,m} := \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s b_0(u)\mathrm{d}u\right\} s\mathrm{d}s < +\infty \tag{10}$$

where $b_0(r)$ is the dissipativity rate of the drift of one particle in the system (3) at distance $r > 0$:

$$b_0(r) = \sup_{x,y,z\in\mathbb{R}^d:|x-y|=r} -\langle \frac{x-y}{|x-y|}, (\nabla V(x) - \nabla V(y)) + (\nabla_x W(x,z) - \nabla_x W(y,z))\rangle. \tag{11}$$

This last condition, taken from [27], is of course reminiscent of the work of Eberle [14, 15] without the interaction potential for convergence to equilibrium in L^1-Wasserstein distance. However in their work the interaction potential is seen only as a perturbation.

2.2. **Uniform Poincaré inequality for mean-field $\mu^{(N)}$.** In the sequel we shall use the notation $\nabla^2_{x_i,x_j} H$ for a C^2-function H on $(\mathbb{R}^d)^N$, defined by

$$\nabla^2_{x_i,x_j} H := (\partial^2_{x_{ik}x_{jl}} H)_{1\le k,l\le d}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{id}) \in \mathbb{R}^d$. Let

$$\lambda_{1,m} = \inf_{N\ge 2}\inf_{1\le i\le N} \lambda_1(\mu_i) \tag{12}$$

where $\lambda_1(\mu_i)$ is the spectral gap of the conditional distribution $\mu_i = \mu_i(dx_i|x^{\hat{i}})$ of x_i knowing $x^{\hat{i}} = (x_j)_{j\neq i}$, i.e. the best constant such that the following Poincaré inequality

$$\lambda_1(\mu_i)\mathrm{Var}_{\mu_i}(f) \le \int_{\mathbb{R}^d} |\nabla_i f|^2 d\mu_i,\ \forall f \in C_b^1(\mathbb{R}^d)$$

holds.

Theorem 1. *In the framework described above, we have always*

$$\lambda_{1,m} \geq \frac{1}{c_{Lip,m}}. \tag{13}$$

Assume that there is some constant $h > -\lambda_{1,m}$ such that for any $(x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$,

$$\frac{1}{N-1}(1_{i\neq j}\nabla^2_{x,y} W(x_i,x_j))_{1\le i,j\le N} \geq hI_{dN} \tag{14}$$

in the order of definite nonnegativity for symmetric matrices, where I_n is the identity matrix of taille n. Then $\mu^{(N)}$ satisfies the following Poincaré inequality

$$(\lambda_{1,m} + h)\,\mathrm{Var}_{\mu^{(N)}}(f) \le \int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)},\ \ f \in C_b^1(\mathbb{R}^{dN}) \tag{15}$$

or equivalently the spectral gap $\lambda_1(\mu^{(N)})$ of $\mathcal{L}^{(N)}$ on $L^2(\mu^{(N)})$, defined as the infimum of those spectral points $\lambda > 0$ of $\mathcal{L}^{(N)}$ on $L^2(\mu^{(N)})$, verifies

$$\lambda_1(\mu^{(N)}) \ge \lambda_{1,m} + h \ge \frac{1}{c_{Lip,m}} + h. \tag{16}$$

Its proof will be given in §3.
The uniform Poincaré inequality in Theorem 1 gives us the following explicit correlation inequality. For any C^1-function f on $\mathbb{R}^d$, denote $\|f\|_{\mathrm{Lip}}^2$ by its Lipschitzian norm w.r.t. the Euclidean metric on $\mathbb{R}^d$.

Corollary 2. *Under the conditions of Theorem 1, for any two bounded Lipschitzian functions f, g on $\mathbb{R}^d$ and $i \neq j$*

$$\mathrm{Cov}_{\mu^{(N)}}(f(x_i), g(x_j)) \le \frac{c_{\mathrm{Lip,m}}}{(1 + c_{\mathrm{Lip,m}} h)(N-1)} \left(\|f\|_{\mathrm{Lip}}^2 + \|g\|_{\mathrm{Lip}}^2 \right) \tag{17}$$

where $\mathrm{Cov}_{\mu^{(N)}}(\cdot,\cdot)$ denotes the covariance of two functions under the probability measure $\mu^{(N)}$. Roughly speaking, two particles x_i and x_j become asymptotically independent at the rate $1/N$.

Proof. The l.h.s of (17) does not depend on (i, j). Applying the Poincaré inequality to $F := \frac{1}{\sqrt{N}} \sum_{i=1}^N f(x_i)$, we have

$$\begin{aligned}\mathrm{Var}_{\mu^{(N)}}(F) &= \mathrm{Var}_{\mu^{(N)}}(f(x_1)) + (N-1)\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) \\ &\le \frac{1}{\lambda_1(\mu^{(N)})} \int |\nabla F|^2 d\mu^{(N)} \le \frac{1}{\lambda_1(\mu^{(N)})} \|f\|_{\mathrm{Lip}}^2,\end{aligned}$$

and therefore

$$\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) \le \frac{1}{(N-1)\lambda_1(\mu^{(N)})} \|f\|_{\mathrm{Lip}}^2.$$

On the other hand, by the first equality above

$$\begin{aligned}\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2)) &= \frac{1}{N-1} \Big(\mathrm{Var}_{\mu^{(N)}}(F) - \mathrm{Var}_{\mu^{(N)}}(f(x_1)) \Big) \\ &\ge -\frac{1}{N-1} \mathrm{Var}_{\mu^{(N)}}(f(x_1)) \ge -\frac{1}{(N-1)\lambda_1(\mu^{(N)})} \|f\|_{\mathrm{Lip}}^2.\end{aligned}$$

Hence we get

$$|\mathrm{Cov}_{\mu^{(N)}}(f(x_1), f(x_2))| \le \frac{1}{(N-1)\lambda_1(\mu^{(N)})} \|f\|_{\mathrm{Lip}}^2 \le \frac{c_{\mathrm{Lip,m}}}{(1 + c_{\mathrm{Lip,m}} h)(N-1)} \|f\|_{\mathrm{Lip}}^2, \tag{18}$$

where the last inequality follows by (16).
Using (18), we obtain

$$\begin{aligned}\mathrm{Cov}_{\mu^{(N)}}(f(x_1),g(x_2)) &= \frac{1}{4}\Big[\mathrm{Cov}_{\mu^{(N)}}((f+g)(x_1),(f+g)(x_2)) - \mathrm{Cov}_{\mu^{(N)}}((f-g)(x_1),(f-g)(x_2))\Big]\\ &\le \frac{c_{\mathrm{Lip,m}}}{4(1+c_{\mathrm{Lip,m}}h)(N-1)}\left(\|f+g\|_{\mathrm{Lip}}^2+\|f-g\|_{\mathrm{Lip}}^2\right)\\ &\le \frac{c_{\mathrm{Lip,m}}}{(1+c_{\mathrm{Lip,m}}h)(N-1)}\left(\|f\|_{\mathrm{Lip}}^2+\|g\|_{\mathrm{Lip}}^2\right)\end{aligned}$$

the desired (17). □

Remark 3. The Poincaré inequality (15) is sharp. In fact, let $d=1$, $V(x)=x^2/2$, $W(x,y)=\beta xy$. In that case $b_0(r)=-r$ (such W does not change b_0), $1/c_{Lip,m}=1=\lambda_{1,m}$. Note that $\lambda_0 := \min\left\{1+\beta,\ 1-\dfrac{\beta}{N-1}\right\}$ is the smallest eigenvalue of the symmetric matrix

$$\frac{1}{N-1}(\beta 1_{i\neq j}) + I_N = \frac{1}{N-1}(\beta 1_{i\neq j}) + \lambda_{1,m}I_N.$$

Our condition (14) for the Poincaré inequality becomes

$$\lambda_0>0.$$

This is necessary even for well defining $\mu^{(N)}$. And our estimate (16) says that $\lambda_1(\mu^{(N)})\ge\lambda_0$. As the matrix of the l.h.s. above is exactly the inverse of the covariance matrix of the centered gaussian distribution $\mu^{(N)}$, its spectral gap is exactly λ_0, showing so the sharpness of this theorem.

Remark 4. Here we give an explicit estimate of $c_{Lip,m}$ under the following assumptions. Assume there are some constants $c_V, c_1, c_W, c_2\in\mathbb{R}$ and $R\ge 0$ such that

$$\langle \nabla V(x)-\nabla V(y), x-y\rangle \ge c_V|x-y|^2 - c_1|x-y|1_{[|x-y|\le R]} \tag{19}$$

$$\langle \nabla_x W(x,z)-\nabla_x W(y,z), x-y\rangle \ge c_W|x-y|^2 - c_2|x-y|1_{[|x-y|\le R]}; \tag{20}$$

for all $x,y\in\mathbb{R}^d$, and $c_V+c_W>0$, then we have for any $r>0$,

$$\begin{aligned} b_0(r) &= \sup_{|x-y|=r,z}\langle\frac{x-y}{|x-y|}, -[(\nabla V(x)-\nabla V(y))+(\nabla_x W(x,z)-\nabla_x W(y,z))]\rangle\\ &\le -(c_V+c_W)r+(c_1+c_2)1_{[r\le R]}\end{aligned}$$

which implies that

$$\begin{aligned} c_{Lip,m} &\le \frac{1}{4}\int_0^\infty \exp\left\{\frac{1}{4}\int_0^s[-(c_V+c_W)u+(c_1+c_2)1_{[0,R]}(u)]\mathrm{d}u\right\}s\mathrm{d}s\\ &\le \frac{1}{4}\int_0^\infty \exp\left\{-\frac{1}{8}(c_V+c_W)s^2+\frac{1}{4}(c_1+c_2)R]\right\}s\mathrm{d}s\\ &= \frac{1}{c_V+c_W}\exp\left(\frac{1}{4}(c_1+c_2)R\right).\end{aligned}$$

Example 1. (Curie-Weiss model) Let $d=1$, $V(x)=\beta(x^4/4-x^2/2)$, $W(x,y)=-\beta Kxy$ where $\beta>0$ is the inverse temperature, $K\in\mathbb{R}^*$. This model is ferromagnetic or anti-ferromagnetic according to $K>0$ or $K<0$.

For this example, we find by elementary analysis

$$b_0(r) = -2V'(r/2) = -2\beta(r^3/8 - r/2),\ r > 0.$$

then

$$\begin{aligned} c_{Lip,m} &= \frac{1}{4}\int_0^\infty \exp\left\{\frac{\beta}{4}\int_0^s (r - \frac{r^3}{4})\mathrm{d}r\right\} s\mathrm{d}s \\ &= \frac{1}{4}\int_0^\infty \exp\left\{\frac{\beta}{4}(\frac{s^2}{2} - \frac{s^4}{16})\right\} s\mathrm{d}s \\ &= e^{\beta/4}\int_0^\infty e^{-\beta(1/2-u)^2}\mathrm{d}u \leq \frac{\sqrt{\pi}}{\sqrt{\beta}}e^{\beta/4} \end{aligned}$$

Let $\lambda(\beta) = \frac{1}{c_{Lip,m}}$. By Theorem 1, if there exists $h > -\lambda(\beta)$ such that

$$-\frac{\beta K}{N-1}(1_{i\neq j}) \geq hI_N$$

then $\lambda_1(\mu^{(N)}) \geq h + \lambda(\beta)$. Note that $(1_{i\neq j})$ has two eigenvalues, $N-1$ and -1. Hence

$$-\frac{\beta K}{N-1}(1_{i\neq j}) \geq \begin{cases} \frac{\beta K}{N-1} I_N, & \text{if } K < 0, \\ -\beta K I_N, & \text{if } K > 0. \end{cases}$$

So taking

$$h = \begin{cases} \frac{\beta K}{N-1}, & \text{if } K < 0, \\ -\beta K, & \text{if } K > 0 \end{cases}$$

we get by Theorem 1,

$$\lambda_1(\mu^{(N)}) \geq \begin{cases} \frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta/4} + \frac{\beta K}{N-1}, & \text{if } K < 0, \\ \frac{\sqrt{\beta}}{\sqrt{\pi}}e^{-\beta/4} - \beta K, & \text{if } K > 0. \end{cases} \tag{21}$$

(It holds automatically if the right hand side above is ≤ 0.)
In particular in the anti-ferromagnetic case (i.e. $K < 0$), for any $\varepsilon > 0$ small enough, $\lambda_1(\mu^{(N)}) \geq \pi^{-1/2}\beta^{1/2}e^{-\beta/4} - \varepsilon > 0$ when the number N of particles is big enough: the mean field should have no phase transition.

Corollary 5. *Assume that $W(x,y) = W_0(x-y)$ where $W_0 : \mathbb{R}^d \to \mathbb{R}$ is C^2, even. If*

1. *∇V is dissipative at infinity in the sense of (19), and*
2. *The Hessian matrix* HessW_0 *of W_0 is bounded from below and from above:*
$$c_W I_d \leq \mathrm{Hess}W_0 \leq C_W I_d \tag{22}$$
and $c_W + c_V > 0$.

Then for all $N \geq 2$,

$$\lambda_1(\mu^{(N)}) \geq \lambda_{1,m} - \frac{N}{N-1}c_W^- - C_W \tag{23}$$

where c_W^- stands for the negative part of c_W.

Remark 6. Let us see what the Bakry-Emery Γ_2-criterion yields. If $\nabla^2 W_0 \geq c_W I_d$ and $\nabla^2 V \geq c_V I_d$, by following the proof of the corollary above, we have $\nabla^2 H \geq (c_V - \frac{N}{N-1} c_W^-) I_{dN}$. Thus by the Bakry-Emery Γ_2-criterion,

$$\lambda_1(\mu^{(N)}) \geq \rho_{LS}(\mu^{(N)}) \geq c_V - \frac{N}{N-1} c_W^-$$

where $\rho_{LS}(\mu^{(N)})$ is the log-Sobolev constant, given in the next subsection.

Remark 7. We notice that if V is super-convex at infinity (i.e. the minimal eigenvalue of $\nabla^2 V(x)$ tends to $+\infty$ when $|x| \to \infty$), then c_V can be taken arbitrarily large, so the condition $c_W + c_V > 0$ on the lower bound c_W of Hess W_0 is always satisfied. In particular, if $W_0(x) = \frac{c_W}{2}|x|^2$ with $c_W < 0$ (then concave and $C_W = c_W$), the uniform Poincaré inequality will hold for all big N by (23) since, in this case,

$$\lambda_{1,m} - \frac{N}{N-1} c_W^- - C_W = \lambda_{1,m} + \frac{1}{N-1} c_W.$$

This phenomenon, apparently strange, can be intuitively explained as follows. The confinement potential, being super-convex, pushes strongly all particles towards some bounded domain; and the interaction potential W_0, being concave, pushes every particle far away from others. This creates an equilibrium: the meaning of our spectral gap estimate (23) for the concave potential W_0.

We now present an example for which some much better estimates (than those in Corollary 5) can be obtained.

Example 2. Let $W(x, y) = W_0(x - y)$ where

$$W_0(x) = \int_{\mathbb{R}^d} e^{-\sqrt{-1}\langle x,y\rangle} \mathrm{d}\nu(y) + \frac{c}{2}|x|^2$$

where ν is some bounded symmetric (i.e. $\nu(-A) = \nu(A)$ for any Borel subset A of $\mathbb{R}^d$) positive measure on $\mathbb{R}^d$ with finite second moment. Let $\Gamma_\nu = (\int y_k y_l d\nu(y))_{1\leq k,l\leq d}$ be the covariance matrix of ν, and $\lambda_{\max}(\Gamma_\nu)$ (resp. $\lambda_{\min}(\Gamma_\nu)$) its maximal (resp. minimal) eigenvalue.
In §4, we will show the following better result :

$$\lambda_1(\mu^{(N)}) \geq \lambda_{1,m} + \frac{1}{N-1}\left(\min\{c, -c(N-1)\} - \lambda_{\max}(\Gamma_\nu)\right). \tag{24}$$

If $c \leq 0$ (then the interaction potential is concave), this implies that the spectral gap of $\mu^{(N)}$ is always uniformly lower bounded.

2.3. **Uniform log-Sobolev inequality for the mean field $\mu^{(N)}$.** Recall that some nonnegative function $f \in L\log L(\mu)$, its entropy w.r.t. the probability measure μ is defined by

$$\mathrm{Ent}_\mu(f) := \int f \log f d\mu - \mu(f)\log\mu(f), \ \ \mu(f) := \int f d\mu.$$

Theorem 8. *Assume that*

(1) *for some best constant $\rho_{\mathrm{LS,m}} > 0$, the conditional marginal distributions $\mu_i := \mu_i(dx_i|x^{\hat{i}})$ on $\mathbb{R}^d$ satisfy the log-Sobolev inequality :*

$$\rho_{\mathrm{LS,m}}\, \mathrm{Ent}_{\mu_i}(f^2) \leq 2\int |\nabla f|^2 \mathrm{d}\mu_i, \ \ f \in C_b^1(\mathbb{R}^d) \tag{25}$$

for all i and $x^{\hat{i}}$;

(2) *(a translation of Zegarlinski's condition)*

$$\gamma_0 = c_{Lip,m} \sup_{x,y\in\mathbb{R}^d,|z|=1} |\nabla^2_{x,y} W(x,y)z| < 1. \tag{26}$$

then $\mu^{(N)}$ satisfies

$$\rho_{\mathrm{LS,m}}(1-\gamma_0)^2 \operatorname{Ent}_{\mu^{(N)}}(f^2) \le 2\int_{(\mathbb{R}^d)^N} |\nabla f|^2 d\mu^{(N)},\ f \in C_b^1((\mathbb{R}^d)^N)$$

i.e. the log-Sobolev constant of $\mu^{(N)}$ verifies

$$\rho_{\mathrm{LS}}(\mu^{(N)}) \ge \rho_{\mathrm{LS,m}}(1-\gamma_0)^2. \tag{27}$$

Remark 9. In this remark we present one approach to establish the first assumption in Theorem 8. Suppose that $\nabla_x^2 W \ge -K_0 I_d$ and V is super-convex in the sense that for any $K > 0$ there exists $R > 0$ such that

$$\nabla^2 V(x) \ge K I_d,\ \text{for } |x| \ge R$$

then V can be decomposed as the sum of a uniform convex function V_c and a bounded function V_b such that

$$\nabla^2 V_c \ge (K_1 + K_0) I_d,$$

therefore, thanks to Bakry-Emery criterion, the probability measure

$$\frac{1}{\tilde{Z}} \exp\left(-V_c(x_i) - \frac{1}{N-1}\sum_{j:j\neq i} W(x_i,x_j)\right) \mathrm{d}x_i$$

satisfies a log-Sobolev inequality with constant K_1. By the bounded perturbation theorem, the conditional measures $\mu_i = \mu_i(\cdot|x^{\hat{i}}), i = 1,\cdots,N$ satisfy a log-Sobolev inequality with a uniform constant $\rho_{LS,m} \ge K_1 \exp(-(\sup V_b - \inf V_b))$ which does not depend on i, x, N.

Example 3. Let us go back to the Curie-Weiss example in dimension 1: $d = 1$, $V(x) = \beta(x^4/4 - x^2/2)$, $W(x,y) = -\beta Kxy$ where $\beta > 0$. As given before we have

$$c_{Lip,m} \le \sqrt{\frac{\pi}{\beta}} e^{\beta/4}.$$

So that

$$\gamma_0 \le c_{Lip,m} \|\nabla^2_{x,y} W\|_\infty \le \sqrt{\pi\beta} e^{\beta/4} |K|$$

which will be smaller than 1 if β or K is sufficiently small.

2.4. **Exponential convergence of McKean-Vlasov equation in entropy and in the Wasserstein metric W_2.** We present now an application of the uniform log-Sobolev inequality in Theorem 8 to the non-linear McKean-Vlasov equation.

Recall at first the relative entropy of a probability measure ν w.r.t. the given probability measure μ on $\mathbb{R}^d$:

$$H(\nu|\mu) := \begin{cases} \int f \log f d\mu = \operatorname{Ent}_\mu(f), & \text{if } \nu \ll \mu, f := \frac{d\nu}{d\mu} \\ +\infty, & \text{otherwise.} \end{cases} \tag{28}$$

The L^p-Wasserstein distance $W_p(\nu,\mu)$ is defined by

$$W_p(\mu,\nu)=\inf_{(X,Y)}\left(\mathbb{E}|X-Y|^p\right)^{1/p}$$

where the infimum is taken over all couples (X,Y) of random variables defined on some probability space, such that the laws of X,Y are respectively μ,ν (a such couple as well as their joint law is called a *coupling of* (μ,ν)). Recall that the space $\mathcal{M}_1^p(\mathbb{R}^d)$ of probability measures with finite p-moment, equipped with L^p-Wasserstein distance W_p, is complete and separable (Villani [26]).

The Fisher-Donsker-Varadhan's information of ν w.r.t. μ is defined by

$$I(\nu|\mu):=\begin{cases}\int|\nabla\sqrt{f}|^2d\mu, & \text{if } \nu\ll\mu, \sqrt{f}:=\sqrt{\frac{d\nu}{d\mu}}\in H^1_\mu\\ +\infty, & \text{otherwise.}\end{cases}\tag{29}$$

where H^1_μ is the domain of the Dirichlet form $\mathcal{E}_\mu[g]=\int|\nabla g|^2d\mu$ (well defined if μ has C^1-density w.r.t. dx). Recall that the log-Sobolev inequality for $\mu^{(N)}$ can be rewritten in

$$\rho_{LS}(\mu^{(N)})H(\nu|\mu^{(N)})\le 2I(\nu|\mu^{(N)}),\ \nu\in\mathcal{M}_1((\mathbb{R}^d)^N).\tag{30}$$

What replaces the role of the relative entropy in interacting particle system for the nonlinear McKean-Vlasov equation is the free energy of a probability measure ν on $\mathbb{R}^d$:

$$E_f(\nu):=\begin{cases}H(\nu|\alpha)+\frac{1}{2}\iint W(x,y)d\nu(x)d\nu(y), & \text{if } H(\nu|\alpha)<+\infty\\ +\infty & \text{otherwise}\end{cases}\tag{31}$$

or more precisely the corresponding mean field entropy

$$H_W(\nu):=E_f(\nu)-\inf_{\tilde\nu\in\mathcal{M}_1(\mathbb{R}^d)}E_f(\tilde\nu).\tag{32}$$

And the substituter of the Fisher-Donsker-Varadhan's information is: if $\nu=f(x)dx$, $\int|x|^2d\nu(x)<+\infty$ and $\nabla f\in L^1_{loc}(\mathbb{R}^d)$ in the distribution sense,

$$I_W(\nu):=\frac{1}{4}\int\left|\frac{\nabla f(x)}{f(x)}+\nabla V(x)+(\nabla_x W\circledast\nu)(x)\right|^2d\nu(x),\tag{33}$$

and $+\infty$ otherwise. Those two objects appeared both in Carrillo-McCann-Villani [10]. The following result generalizes the main result of [10] from the convex framework to the more general non-convex case.

Theorem 10. *Assume the uniform marginal log-Sobolev inequality, i.e.* (25) *with* $\rho_{LS,m}>0$, *and the uniqueness condition of Zegarlinski (26). Then*

(1) *There exists a unique minimizer* ν_∞ *of* H_W *over* $\mathcal{M}_1(\mathbb{R}^d)$*;*
(2) *The following (nonlinear) log-Sobolev inequality*
$$\rho_{LS}H_W(\nu)\le 2I_W(\nu),\ \nu\in\mathcal{M}_1(\mathbb{R}^d)\tag{34}$$
holds, where
$$\rho_{LS}:=\limsup_{N\to\infty}\rho_{LS}(\mu^{(N)})\ge\rho_{LS,m}(1-\gamma_0)^2.$$
(3) *The following Talagrand's transportation inequality holds*
$$\rho_{LS}W_2^2(\nu,\nu_\infty)\le 2H_W(\nu),\ \nu\in\mathcal{M}_1(\mathbb{R}^d)\tag{35}$$

(4) *For the solution ν_t of the McKean-Vlasov equation with the given initial distribution ν_0 of finite second moment,*

$$H_W(\nu_t) \le e^{-t\cdot\rho_{LS}/2} H_W(\nu_0),\ t \ge 0 \tag{36}$$

and in particular

$$W_2^2(\nu_t, \nu_\infty) \le \frac{2}{\rho_{LS}} e^{-t\cdot\rho_{LS}/2} H_W(\nu_0),\ t \ge 0 \tag{37}$$

Remark 11. In the work by Carrillo-McCann-Villani [10], presuming the presence of confining potential, such results were obtained in the case where $W(x, y) = W_0(x - y)$ and

(1) either $\nabla^2 V > ||(\nabla^2 W)^-||_{L^\infty}$ (in particular, V is uniformly strictly convex);
(2) or W is strictly convex at infinity, and both V and W are strictly convex (possibly degenerate at the origin).

In particular, V was required to be convex in both situations. If we consider the case in dimension one, $V(x) = \beta(x^4/4 - x^2/2)$ and $W_0(x) = -\beta K x^2/2$ with $K \ge 0$. Then by analogous calculations than for the Curie-Weiss model, we have $c_{Lip,m} \le \sqrt{\pi/\beta} e^{\beta(1+K)^2/4}$ so that $\gamma_0 \le \sqrt{\pi\beta} K e^{\beta(1+K)^2/4}$ and thus the conditions (25), (26) are verified for β or K small enough for example, cases not covered in [10]. Our conditions are quite comparable with the results obtained in [15] but they only consider convergence in L^1-Wasserstein distance. Remark also that the conditions are comparable to the assumptions made in [13] to get an uniform in time propagation of chaos (but in L^1-Wasserstein distance) which explains why we may pass to the limit in the number of particles.

3. Lipschitzian spectral gap for conditional distribution

Notice that the conditional distribution $\mu_i(dx_i) := \mu_i(dx_i | x_j, j \ne i)$ of x_i knowing $x^{\hat{i}} := (x_j)_{j\ne i}$ of our mean field measure $\mu^{(N)}$ defined in (5) is given by

$$\mathrm{d}\mu_i(x_i) = \frac{1}{Z_i} \exp\left\{ -V(x_i) - \frac{1}{N-1} \sum_{j:\ j\ne i} W(x_i, x_j) \right\} \mathrm{d}x_i$$

where $Z_i = Z_i(x^{\hat{i}})$ is the normalization factor. Let

$$H_i(x_i) := V(x_i) + \frac{1}{N-1} \sum_{j:\ j\ne i} W(x_i, x_j)$$

be the potential associated with μ_i. The generator $\mathcal{L}_i^{(N)} = \Delta_i - \nabla_i H_i \cdot \nabla_i$ given in (4), with $(x_j)_{j\ne i}$ fixed, is symmetric w.r.t. μ_i. By the definition (11) of $b_0(r)$, for all $x, y \in (\mathbb{R}^d)^N$,

$$\begin{aligned}
&\langle \frac{x_i - y_i}{|x_i - y_i|}, -[\nabla_i H(x) - \nabla_i H(x^{\hat{i}, y_i})] \rangle \\
&= \frac{1}{N-1} \sum_{j\ne i} \langle \frac{x_i - y_i}{|x_i - y_i|}, -[(\nabla V(x_i) + \nabla_x W(x_i, x_j)) - (\nabla V(y_i) + \nabla_x W(y_i, x_j)] \rangle \\
&\le b_0(|x_i - y_i|)
\end{aligned}$$

where $x^{\hat{i}, y_i} \in (\mathbb{R}^d)^N$ is given by $(x^{\hat{i}, y_i})_j = x_j, j \ne i$, $(x^{\hat{i}, y_i})_i = y_i$. So we have the following result (due to the third named author [27]), which is the starting point of our investigation.

Lemma 12. *Assume (10). Then the Poisson operator $(-\mathcal{L}_i)^{-1}$ on the Banach space $C_{\mathrm{Lip},0}(\mathbb{R}^d)$ of Lipschitzian continuous functions f on $\mathbb{R}^d$ with $\mu_i(f) = 0$, equipped with the norm $\|f\|_{\mathrm{Lip}}$, is bounded and its norm*

$$||(-\mathcal{L}_i)^{-1}||_{Lip} \le c_{Lip,m} \tag{38}$$

where $c_{Lip,m}$ is given in (10). In particular the spectral gap $\lambda_1(\mu_i)$ of $\mathcal{L}_i$ on $L^2(\mu_i)$ satisfies

$$\lambda_1(\mu_i) \ge \frac{1}{c_{Lip,m}}. \tag{39}$$

4. Uniform Poincaré inequality : proof of Theorem 1

Let $V \in C^2(\mathbb{R}^d)$ be the confinement potential, U a C^2-potential of interaction on $(\mathbb{R}^d)^N$ and $H(x_1, \cdots, x_N) = \sum_{i=1}^N V(x_i) + U(x_1, \cdots, x_N)$ the Hamiltonian. Now consider the probability measure

$$d\mu := \frac{1}{Z} e^{-H} dx_1 \cdots dx_N$$

where $Z = \int_{(\mathbb{R}^d)^N} e^{-H(x)} dx$ is the normalization constant (called often *partition function*), assumed to be finite. We denote by $\mu_i = \mu(dx_i | x^{\hat{i}})$ the conditional distribution of x_i given $x^{\hat{i}} := (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N)$ under μ. It is given by

$$\mu_i(\mathrm{d}x_i) = \frac{1}{Z_i} e^{-U(x)-V(x_i)} \mathrm{d}x_i, \;\; Z_i = Z_i(x^{\hat{i}}) := \int e^{-U(x)-V(x_i)} \mathrm{d}x_i < +\infty \text{ (assumed).}$$

We shall describe below conditions on the Hamiltonian H such that μ satisfies a Poincaré inequality, namely for some positive constant ρ,

$$\rho \int f^2 \mathrm{d}\mu \le \int |\nabla f|^2 \mathrm{d}\mu$$

for every smooth function $f \in C_b^1((\mathbb{R}^d)^N)$. The largest ρ is called the spectral gap of μ, denoted as $\lambda_1(\mu)$.

Proposition 13. *Assume that $Z = \int_{(\mathbb{R}^d)^N} e^{-H(x)} dx < +\infty, Z_i(x^{\hat{i}}) < +\infty$ for all $i, x^{\hat{i}}$. If*

(1) *the marginal conditional distributions μ_i satisfy the uniform Poincaré inequality, i.e.*

$$\lambda_{1,m} := \inf_{1 \le i \le N, x^{\hat{i}} \in (\mathbb{R}^d)^{N-1}} \lambda_1(\mu_i) > 0, \tag{40}$$

(2) *for some constant $h \in \mathbb{R}$,*

$$(1_{i \ne j} \nabla^2_{x_i, x_j} U) \ge h I_{dN}, \tag{41}$$

in the sense of nonnegative definiteness of symmetric matrices;

then

$$\lambda_1(\mu) \ge h + \lambda_{1,m}.$$

This result is essentially due to Ledoux [18]. Indeed, in the case of $d = 1$, if $\mathrm{Hess}\,(U) \ge \underline{\lambda} \mathrm{I}_N$ and $\partial_{ii} U(x) \le \bar{\lambda}$ for all i and $x^{\hat{i}}$, then for every $v = (v_1, v_2, \cdots, v_N) \in \mathbb{R}^N$,

$$\sum_{i \ne j} v_i \partial^2_{ij} U v_j = \langle \mathrm{Hess}\,(U) v, v \rangle - \sum_i v_i^2 \partial^2_{ii} U \ge (\underline{h} - \bar{h}) |v|^2$$

i.e. the assumption (41) holds with $h = \underline{\lambda} - \overline{\lambda}$. This proposition gives $\lambda_1(\mu) \geq \lambda_{1,m} + \underline{\lambda} - \overline{\lambda}$, which is the original result of Ledoux [18].

For the convenience of the reader, we reproduce the beautiful proof of Ledoux [18, Prop. 3.1].

Proof. Of course we may and will assume that $\lambda_{1,m} + h > 0$. Let $\mathcal{L} = \Delta - \nabla H \cdot \nabla$ be the symmetric generator associated with the probability measure μ. By the dual description of Poincaré inequality [3, Prop. 4.8.3], the conclusion above is equivalent to

$$\int (\mathcal{L}f)^2 \mathrm{d}\mu \geq (\lambda_{1,m} + h) \int |\nabla f|^2 \mathrm{d}\mu.$$

Thanks to the Bakry-Emery's formula $\int \Gamma_2(f)\mathrm{d}\mu = \int (\mathcal{L}f)^2 \mathrm{d}\mu$ and

$$\Gamma_2(f) = \|\nabla^2 f\|_{\mathrm{HS}}^2 + \langle \nabla^2 H \nabla f, \nabla f \rangle$$

where $||A||_{\mathrm{HS}} := (\sum_{i,j} |a_{ij}|^2)^{1/2}$ is the Hilbert-Schmidt norm of a matrix $A = (a_{ij})$, we have

$$\begin{aligned}
\int (\mathcal{L}f)^2 \mathrm{d}\mu &= \int \left(||\nabla^2 f||_{\mathrm{HS}}^2 + \langle \nabla^2 H \nabla f, \nabla f \rangle \right) \mathrm{d}\mu \\
&= \int \left(||\nabla^2 f||_{\mathrm{HS}}^2 + \sum_{i=1}^n \langle \mathrm{Hess}\,(V)(x_i) \nabla_{x_i} f, \nabla_{x_i} f \rangle + \langle \mathrm{Hess}\,(U) \nabla f, \nabla f \rangle \right) \mathrm{d}\mu \\
&\geq \sum_{1 \leq i \leq N} \int \int_{\mathbb{R}^d} \left(||\nabla_{x_i}^2 f||_{\mathrm{HS}}^2 + \langle (\mathrm{Hess}\,(V)(x_i) + \nabla_{x_i,x_i}^2 U) \nabla_{x_i} f, \nabla_{x_i} f \rangle \right) \mathrm{d}\mu_i d\mu \\
&\quad + \int \sum_{i \neq j} \langle \nabla_{x_i,x_j}^2 U \nabla_{x_i} f, \nabla_{x_j} f \rangle \mathrm{d}\mu
\end{aligned}$$

Applying the above characterization of the Poincaré inequality but to the conditional measures μ_i, we have

$$\int \left[||\nabla_{x_i}^2 f||_{\mathrm{HS}}^2 + \langle (\mathrm{Hess}\,(V)(x_i) + \nabla_{x_i,x_i}^2 U) \nabla_{x_i} f, \nabla_{x_i} f \rangle \right] \mathrm{d}\mu_i \geq \lambda_{1,m} \int |\nabla_{x_i} f|^2 \mathrm{d}\mu_i$$

for any i and any given $x^{\hat{i}}$. Moreover by the assumption (41),

$$\int \sum_{i \neq j} \langle \nabla_{x_i x_j}^2 U \nabla_{x_i} f, \nabla_{x_j} f \rangle \mathrm{d}\mu \geq h \int |\nabla f|^2 \mathrm{d}\mu.$$

This, combined with the previous inequality, yields the desired inequality. □

We come back to the mean field setting.

Proof of Theorem 1. We shall apply Proposition 13 to $\mu = \mu^{(N)}$. With the notations above, the interaction potential U is then given by

$$U(x) = \frac{1}{N-1} \sum_{1 \leq i < j \leq N} W(x_i, x_j) = \frac{1}{2} \sum_{i=1}^N U_i(x) \tag{42}$$

where $U_i(x) = \frac{1}{N-1} \sum_{j: j \neq i} W(x_i, x_j)$. For $i \neq j$,

$$\nabla_{x_i,x_j}^2 U = \frac{1}{N-1} (\nabla_{x,y}^2 W)(x_i, x_j)$$

therefore the assumption (14) implies the condition (41) with constant h in Proposition 13. On the other hand, since $\mu_i(dx_i|x^{\hat{i}}) = e^{-[V(x_i)+U_i(x)]}dx_i/Z_i(x^{\hat{i}})$ and

$$-\langle \frac{x_i - y_i}{|x_i - y_i|}, \nabla_{x_i}[V(x_i) + U_i(x)] - \nabla_{x_i}[V(y_i) + U_i(x^{\hat{i},y_i})]\rangle \le b_0(|x_i - y_i|)$$

as noted in §3, thanks to the assumption (10), Lemma 12 yields $\lambda_1(\mu_i) \ge 1/c_{\mathrm{Lip,m}}$. Hence we can apply Proposition 13 to the invariant measure $\mu^{(N)}$, and obtain (16). □

Proof of Corollary 5. In this particular context $W(x,y) = W_0(x-y)$, for $U(x)$ given by (42),

$$\nabla^2_{x_i,x_i} U(x) = \frac{1}{N-1}\sum_{j\ne i}(\nabla^2 W_0)(x_i - x_j);\ \nabla^2_{x_i,x_j} U = -\frac{1}{N-1}(\nabla^2 W_0)(x_i - x_j) \text{ for } i \ne j$$

i.e. $\nabla^2 U = -\frac{1}{N-1}(A_{ij})$ where $A_{ij} = (\nabla^2 W_0)(x_i - x_j)$ for $i \ne j$ and $A_{ii} = -\sum_{j:j\ne i} A_{ij}$. As A_{ij} is symmetric and $A_{ij} = A_{ji}$, we have for any $u = (u_1, \cdots, u_N)$ in $(\mathbb{R}^d)^N$,

$$\begin{aligned}
-\sum_{i,j}\langle u_i, A_{ij}u_j\rangle &= \sum_{i\ne j}\langle -u_i, A_{ij}(u_j - u_i)\rangle = \sum_{i\ne j}\langle u_j, A_{ij}(u_j - u_i)\rangle \\
&= \frac{1}{2}\sum_{i\ne j}\langle (u_j - u_i), A_{ij}(u_j - u_i)\rangle \\
&\ge \frac{c_W}{2}\sum_{i\ne j}|u_j - u_i|^2 = c_W\sum_{i,j}\langle u_j, u_j - u_i\rangle \text{ (by the previous equality with } A_{ij} = I) \\
&= c_W N\left(|u|^2 - N|\bar{u}|^2\right) = c_W N|u - \bar{u}|^2 \\
&\ge \begin{cases} c_W N|u|^2, & \text{if } c_W \le 0. \\ 0 & \text{if } c_W > 0 \end{cases}
\end{aligned}$$

Therefore $\nabla^2 U \ge -c_W^- \frac{N}{N-1} I_{dN}$. Obviously $\nabla^2_{x_i,x_i} U \le C_W I_d$. Then

$$(1_{i\ne j}\nabla^2_{x_i,x_j} U) = \nabla^2 U - (1_{i=j}\nabla^2_{x_i,x_i} U) \ge -\left(c_W^- \frac{N}{N-1} + C_W\right) I_{dN}$$

It remains to apply Proposition 13 to get the desired spectral gap estimate (23).

□

Proof of (24) in Example 2. Notice that

$$\begin{aligned}
(1_{i\ne j}\nabla^2_{x_i,x_j} W(x_i,x_j)) &= \left(1_{i\ne j}[-cI_d + \int e^{-\sqrt{-1}(x_i - x_j)\cdot y} yy^T \mathrm{d}\nu(y)]\right) \\
&= -c(1_{i\ne j} I_d) + \left(\int e^{-\sqrt{-1}(x_i - x_j)\cdot y} yy^T \mathrm{d}\nu(y)\right) - (1_{i=j}\int yy^T \mathrm{d}\nu(y)) \\
&\ge cP_{\mathbf{H}} - c(N-1)P_{\mathbf{H}^\perp} - \lambda_{\max}(\Gamma_\nu) I_{dN}
\end{aligned}$$

where the second expression in the second line is a positive-definite matrix, and $P_{\mathbf{H}}, P_{\mathbf{H}^\perp}$ are respectively the orthogonal projection from $(\mathbb{R}^d)^N$ to $\mathbf{H}$ and to its orthogonal complement

$\mathbf{H}^\perp$,

$$\mathbf{H} = \{x = (x_1, \cdots, x_N); \bar{x} := \frac{1}{N}\sum_{i=1}^N x_i = 0\},$$

$$\mathbf{H}^\perp = \{x = (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N; x_1 = x_2 = \cdots = x_N\}.$$

Thus we obtain from Theorem 1

$$\lambda_1(\mu^{(N)}) \geq \lambda_{1,m} + \frac{1}{N-1}\left(\min\{c, -c(N-1)\} - \lambda_{\max}(\Gamma_\nu)\right)$$

which is the desired inequality (24). □

5. Uniform log-Sobolev inequality

Inspired by Dobrushin's uniqueness condition for the Gibbs measures, Zegarlinski [31, Theorem 0.1] proved a criterion about the logarithmic Sobolev inequality for the Gibbs measure $\mu = e^{-H}dx/Z$ on $(\mathbb{R}^d)^N$ in terms of the conditional marginal distributions $\mu_i = \mu(dx_i|x^{\hat{i}})$. Let us introduce at first Zegarlinski's dependence coefficient $c_{ij}^{\mathbf{Z}}$ of μ_j upon x_i: this is the best nonnegative constant such that

$$|\nabla_i(\mu_j(f^2))^{1/2}| \leq (\mu_j(|\nabla_i f|^2))^{1/2} + c_{ij}^{\mathbf{Z}}(\mu_j(|\nabla_j f|^2))^{1/2} \tag{43}$$

for all smooth strictly positive functions $f(x_1, \cdots, x_N)$. Obviously $c_{ii}^{\mathbf{Z}} = 0$. The matrix $c^{\mathbf{Z}} := (c_{ij}^{\mathbf{Z}})_{1\leq i,j\leq N}$ will be called Zegarlinski's matrix of interdependence in the sequel.

Theorem 14. **(Zegarlinski** [31, Theorem 0.1]**)** *If*

(1) *μ_i satisfies a uniform log-Sobolev inequality (LSI in short), i.e.*

$$\rho_{\mathrm{LS,m}} := \inf_{1\leq i\leq N, x^{\hat{i}}\in(\mathbb{R}^d)^{N-1}} \rho_{LS}(\mu_i) > 0.$$

(2) *The following* ***Zegarlinski's condition*** *is verified*

$$\gamma := \sup_{1\leq i\leq N} \max\{\sum_{1\leq j\leq N} c_{ji}^{\mathbf{Z}}, \sum_{1\leq j\leq N} c_{ij}^{\mathbf{Z}}\} < 1. \tag{44}$$

Then the Gibbs measure μ satisfies the logarithmic Sobolev inequality

$$\rho_{\mathrm{LS,m}}(1-\gamma)^2 \operatorname{Ent}_\mu(f^2) \leq 2\mu(|\nabla f|^2) \tag{45}$$

for all smooth bounded functions f on $(\mathbb{R}^d)^N$, i.e.

$$\rho_{LS}(\mu) \geq \rho_{\mathrm{LS,m}}(1-\gamma)^2.$$

Our objective is to estimate $c_{ij}^{\mathbf{Z}}$. We begin with a simple observation :

Lemma 15. *If for any function $g = g(x_j) \in C_b^1(\mathbb{R}^d)$ on the single particle x_j,*

$$|\nabla_i \mu_j(g)| \leq c_{ij}\mu_j(|\nabla g|), \tag{46}$$

then $c_{ij}^{\mathbf{Z}} \leq c_{ij}$.

Proof. For any $0 < g \in C_b^1((\mathbb{R}^d)^N)$, by the condition (46), we have for all $i \neq j$,

$$|\nabla_i \sqrt{\mu_j(g)}| = \frac{1}{2\sqrt{\mu_j(g)}}[|\mu_j(\nabla_i g) + (\nabla_{x_i} \int g(x_j, y^{\hat{j}}) \mathrm{d}\mu_j(x_j|x^{\hat{j}}))|_{y^{\hat{j}}=x^{\hat{j}}}|]$$
$$\leq \frac{1}{2\sqrt{\mu_j(g)}}\left[\mu_j(|\nabla_i g|) + c_{ij}\mu_j(|\nabla_j g|)\right].$$

When $g = f^2$ with $f > 0$, we have by the Cauchy-Schwarz inequality for all i, j,

$$\mu_j(|\nabla_i g|) = 2\mu_j(f|\nabla_i f|) \leq 2\sqrt{\mu_j(f^2)\mu_j(|\nabla_i f|^2)}.$$

Substituting it into the previous inequality we get

$$|\nabla_i \sqrt{\mu_j(f^2)}| \leq \sqrt{\mu_j(|\nabla_i f|^2)} + c_{ij}\sqrt{\mu_j(|\nabla_j f|^2)}$$

so it follows $c_{ij}^{\mathbf{Z}} \leq c_{ij}$. □

Lemma 16. *For the mean field Gibbs measure $\mu = \mu^{(N)}$, the interdependence coefficient $c_{ji}^{\mathbf{Z}}$ satisfies*

$$c_{ji}^{\mathbf{Z}} \leq \frac{1}{N-1} c_{\mathrm{Lip,m}} \|\nabla_{x,y}^2 W\|_\infty, \ i \neq j$$

where $c_{\mathrm{Lip,m}}$ is given by (10),

$$\|\nabla_{x,y}^2 W\|_\infty := \sup_{x,y\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^d, |z|=1} |\nabla_{x,y}^2 W(x,y)z|.$$

Proof. For any $z \in \mathbb{R}^d$ with $|z| = 1$ and $g = g(x_i) \in C_0^2(\mathbb{R}^d)$,

$$\nabla_{x_j}\mu_i(g) = \nabla_{x_j}\Big(\int g(x_i) e^{-H(x_1,x_2,\cdots,x_N)} dx_i \Big/ \int e^{-H(x_1,x_2,\cdots,x_N)} dx_i\Big)$$
$$= \frac{\int g(x_i)(-\nabla_{x_j} H) e^{-H} dx_i}{\int e^{-H} dx_i} + \frac{\int g(x_i) e^{-H} dx_i \int \nabla_{x_j} H e^{-H} dx_i}{(\int e^{-H} dx_i)^2}$$
$$= -\int g(x_i)\nabla_{x_j} H d\mu_i + \int g(x_i) d\mu_i \int \nabla_{x_j} H d\mu_i$$
$$= \mathrm{Cov}_{\mu_i}(g, -\nabla_{x_j} H) = \mathrm{Cov}_{\mu_i}(g, -\frac{1}{N-1}(\nabla_y W)(x_i, x_j))$$

and so

$$z \cdot \nabla_{x_j}\mu_i(g) = \mathrm{Cov}_{\mu_i}(g, -\frac{1}{N-1}(\nabla_y W)(x_i, x_j) \cdot z)$$
$$= -\frac{1}{N-1}\langle(-\mathcal{L}_i)g, (-\mathcal{L}_i)^{-1}((\nabla_y W)(\cdot, x_j) \cdot z - \mu_i((\nabla_y W)(\cdot, x_j) \cdot z)\rangle_{\mu_i}$$
$$= -\frac{1}{N-1}\int \nabla_i g \cdot \nabla_i(-\mathcal{L}_i)^{-1}[(\nabla_y W)(\cdot, x_j) \cdot z - \mu_i((\nabla_y W)(\cdot, x_j) \cdot z)] d\mu_i.$$

By Lemma 12,

$$\begin{aligned}&\|\nabla_i(-\mathcal{L}_i)^{-1}((\nabla_y W)(\cdot,x_j)\cdot z-\mu_i((\nabla_y W)(\cdot,x_j)\cdot z))\|_{L^\infty(\mu_i)}\\ &\le c_{\mathrm{Lip,m}}\sup_{x_i,x_j}|\nabla_{x_i}((\nabla_y W)(x_i,x_j)\cdot z)|\\ &= c_{\mathrm{Lip,m}}\sup_{x,y\in\mathbb{R}^d}|\nabla^2_{x,y}W(x,y)z|\\ &\le c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty.\end{aligned}$$

Plugging it into the previous inequality, we obtain

$$|\nabla_{x_j}\mu_i(g)|=\sup_{|z|=1}|z\cdot\nabla_{x_j}\mu_i(g)|\le\frac{1}{N-1}c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty|\mu_i(\nabla_{x_i}g)|$$

which, by Lemma 15, completes the proof. □

Proof of Theorem 8. By Lemma 16,

$$\gamma=\sup_{1\le i\le N}\max\Big\{\sum_{1\le j\le N}c^{\mathbf{Z}}_{ji},\sum_{1\le j\le N}c^{\mathbf{Z}}_{ij}\Big\}\le c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty=\gamma_0<1.$$

Then Theorem 8 follows directly from Theorem 14. □

6. Exponential convergence of McKean-Vlasov equation

Assume that $\mu^{(N)}$ satisfies a uniform log-Sobolev inequality with constant

$$\rho_{LS}=\limsup_{N\to\infty}\rho_{LS}(\mu^{(N)})>0.$$

That is the case if $c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty<1$ by Theorem 8, more precisely

$$\rho_{LS}\ge\rho_{LS,m}(1-c_{\mathrm{Lip,m}}\|\nabla^2_{x,y}W\|_\infty)^2.$$

6.1. **Free energy, entropy related to the McKean-Vlasov equation.** The entropy $H_W(\nu)$ can be identified as the mean relative entropy per particle of $\nu^{\otimes N}$ w.r.t. the mean field Gibbs measure $\mu^{(N)}$:

Lemma 17. *For any probability measure ν on $\mathbb{R}^d$ such that $H(\nu|\alpha)<+\infty$,*

$$\frac{1}{N}H(\nu^{\otimes N}|\mu^{(N)})\to H_W(\nu). \tag{47}$$

Proof. Recall that $\alpha=\frac{1}{C}e^{-V}dx$. By the assumption (H1), it is known that ([12])

$$\int e^{\lambda_0|x|^2}d\alpha(x)<+\infty \text{ for some } \lambda_0>0. \tag{48}$$

Let

$$\tilde Z_N:=\int\exp\left(-\frac{1}{2(N-1)}\sum_{i\ne j}W(x_i,x_j)\right)d\alpha^{\otimes N}$$

so that

$$d\mu^{(N)}=\frac{1}{\tilde Z_N}\exp\left(-\frac{1}{2(N-1)}\sum_{i\ne j}W(x_i,x_j)\right)d\alpha^{\otimes N}.$$

Let $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ such that $H(\nu|\alpha) < +\infty$. Since $H(\nu^{\otimes 2}|\alpha^{\otimes 2}) = 2H(\nu|\alpha) < +\infty$, by Donsker-Varadhan's variational formula of entropy, (48) and the fact that $|W(x,y)| \le C(1+|x|^2+|y|^2)$ (for $\nabla^2 W$ is bounded), we have $W \in L^1(\nu^{\otimes 2})$. Therefore

$$\begin{aligned}\frac{1}{N}H(\nu^{\otimes N}|\mu^{(N)}) &= \frac{1}{N}\int \log \frac{d\nu^{\otimes N}}{d\mu^{(N)}} d\nu^{\otimes N} \\ &= \frac{1}{N}\int \sum_{i=1}^N \log \frac{d\nu}{d\alpha}(x_i) d\nu^{\otimes N} + \int \frac{1}{2N(N-1)} \sum_{i\neq j} W(x_i,x_j) d\nu^{\otimes N} + \frac{1}{N}\log \tilde{Z}_N \\ &= H(\nu|\alpha) + \frac{1}{2}\iint W(x,y)d\nu(x)d\nu(y) + \frac{1}{N}\log \tilde{Z}_N\end{aligned}$$

By [28, (3.30)],

$$\lim_{N\to\infty} \frac{1}{N}\log \tilde{Z}_N = -\inf_{\nu} E_f(\nu).$$

Combining those two equalities we obtain (47). □

The following super-additivity of the relative entropy w.r.t. a product probability measure should be known.

Lemma 18. *Let $\prod_{i=1}^N \alpha_i, Q$ be respectively a product probability measure and a probability measure on $E_1 \times \cdots \times E_N$ where E_i's are Polish spaces, and Q^i the marginal distribution of x_i under Q. Then*

$$H(Q|\prod_{i=1}^N \alpha_i) \ge \sum_{i=1}^N H(Q^i|\alpha_i).$$

Proof. Let $Q_i(\cdot|x_{[1,i-1]})$ be the conditional distribution of x_i knowing $x_{[1,i-1]} = (x_1, \cdots, i-1)$ (knowing nothing if $i=1$). We have

$$\begin{aligned}H(Q|\prod_{i=1}^N \alpha_i) &= \mathbb{E}^Q \log \frac{dQ}{d\prod_{i=1}^N \alpha_i} = \mathbb{E}^Q \sum_{i=1}^N \log \frac{Q_i(dx_i|x_{[1,i-1]})}{\alpha_i(dx_i)} \\ &= \mathbb{E}^Q \sum_{i=1}^n H(Q_i(\cdot|x_{[1,i-1]})|\alpha_i).\end{aligned}$$

Since $\mathbb{E}^Q Q_i(\cdot|x_{[1,i-1]}) = Q^i(\cdot)$, we obtain by the convexity of the relative entropy

$$\mathbb{E}^Q H(Q_i(\cdot|x_{[1,i-1]})|\alpha_i) \ge H(Q^i|\alpha_i)$$

where the desired super-additivity follows. □

Lemma 19. *Let μ be a probability measure on some Polish space S and $U : S \to (-\infty, +\infty]$ a measurable potential satisfying*

$$\int e^{-pU} d\mu < +\infty$$

for some $p > 1$. Consider the Boltzmann probability measure $\mu_U = e^{-U} d\mu/C$. If $H(\nu|\mu_U) < +\infty$, then $H(\nu|\mu) < +\infty$ and $U \in L^1(\nu)$, and

$$H(\nu|\mu_U) = H(\nu|\mu) + \int U d\nu - \log \int e^{-U} d\mu.$$

Proof. For any measurable function f on S, let

$$\Lambda_\mu(f) := \log \int e^f d\mu \in (-\infty, +\infty]$$

be the log-Laplace transform w.r.t. μ, which is convex in f (by Hölder's inequality). Then

$$\Lambda_{\mu_U}(f) = \log \int e^f d\mu_U = \Lambda_\mu(-U + f) - \Lambda_\mu(-U) \leq \frac{1}{p}\Lambda_\mu(-pU) + \frac{1}{q}\Lambda_\mu(qf) - \Lambda_\mu(-U)$$

where $q = p/(p-1)$. By Donsker-Varadhan's variational formula,

$$\begin{aligned} H(\nu|\mu_U) &= \sup_{f \in b\mathcal{B}} \left(\nu(f) - \Lambda_{\mu_U}(f)\right) \\ &\geq \sup_{f \in b\mathcal{B}} \left(\nu(f) - \frac{1}{q}\Lambda_\mu(qf)\right) + \Lambda_\mu(-U) - \frac{1}{p}\Lambda_\mu(-pU) \\ &= \frac{1}{q} H(\nu|\mu) + \Lambda_\mu(-U) - \frac{1}{p}\Lambda_\mu(-pU). \end{aligned}$$

Hence if $H(\nu|\mu_U) < +\infty$, $H(\nu|\mu) < +\infty$ or equivalently $\log \frac{d\nu}{d\mu} \in L^1(\nu)$, and $\log \frac{d\nu}{d\mu_U} = \log \frac{d\nu}{d\mu} + U + \Lambda_\mu(-U) \in L^1(\nu)$. This completes the proof of the Lemma. □

Lemma 20. (propagation of chaos) *Let $(\nu_t)_{t \geq 0}$ be the solution of the McKean-Vlasov equation with the given initial distribution ν_0 such that $\int |x|^2 d\nu_0(x) < +\infty$. Let μ_t^N be the law of $X^N(t) = (X_1^N(t), \cdots, X_N^N(t))$ solving the S.D.E.* (3) *with initial condition $\mu_0^N = \nu_0^{\otimes N}$, and $\mu_t^{N,I}$ the law of the particles $(X_i^N(t))_{i \in I}$ for any index set $I \subset \mathbb{N}^*$. Then for each $t \in \mathbb{R}$ and each finite subset I of $\mathbb{N}^*$, $\mu_t^{N,I} \to \nu_t^{\otimes I}$ in the L^2-Wasserstein metric W_2 as $N \to \infty$.*

This is well known, see [25] or [11].

Lemma 21. (uniqueness of the minimizer of H_W) *If $c_{Lip,m} \|\nabla^2_{xy} W\|_\infty < 1$, then the minimizer ν_∞ of the free energy $E_f(\nu)$ is unique.*

Proof. By [28], under (H2), if $H(\nu|\alpha) < +\infty$, $\iint W^-(x,y) d\nu(x) d\nu(y) < +\infty$ and $E_f : \mathcal{M}_1(\mathbb{R}^d) \to \mathbb{R}$ is inf-compact. Then a minimizer ν_∞ of E_f exists.
If a probability measure ν is a minimizer of E_f, $H(\nu|\alpha) < +\infty$, and then $\int |x|^2 d\nu < +\infty$ by (H1). Regarding the Gateaux-derivative, we see that ν must be a fixed point of the mapping Φ defined by

$$\Phi(\nu) := \frac{1}{Z'} \exp(-V - W \circledast \nu) dx$$

where Z' is the normalizing constant. Here $W \circledast \nu$ is well defined because $|W(x,y)| \leq C(1 + |x|^2 + |y|^2)$ by the boundedness of the second derivatives of W.
We claim that $\Phi : \mathcal{M}_1^2(\mathbb{R}^d) \to \mathcal{M}_1^2(\mathbb{R}^d)$. Indeed, since the hamiltonian $H_\nu = V + W \circledast \nu$ (for any $\nu \in \mathcal{M}_1^2(\mathbb{R}^d)$) satisfies again the dissipative rate condition

$$-\langle \frac{x-y}{|x-y|}, \nabla H_\nu(x) - \nabla H_\nu(y) \rangle \leq b_0(|x-y|), \ x, y \in \mathbb{R}^d$$

(as in §3), the associated generator $\mathcal{L}_\nu = \Delta - \nabla H_\nu \cdot \nabla$ satisfies the Lipschitzian spectral gap estimate (38) by Lemma 12. That implies the spectral gap of $\nu' = \Phi(\nu)$, in particular $\int e^{\delta|x|}d\nu' < +\infty$ for some $\delta > 0$ ([5]). Then if $\nu \in \mathcal{M}_1^2(\mathbb{R}^d)$, $\Phi(\nu) \in \mathcal{M}_1^2(\mathbb{R}^d)$.

Now for the uniqueness of the minimizer of E_f, it remains to show that Φ is contractive on $(\mathcal{M}_1^2(\mathbb{R}^d), W_1)$. Let $\mu_k = \Phi(\nu_k), k = 0, 1$, and

$$\nu_t := (1-t)\nu_0 + t\nu_1,\ \ \mu_t = \Phi(\nu_t).$$

For any 1-Lipschitzian function f, we have

$$\begin{aligned}\frac{d}{dt}\mu_t(f) &= \mathrm{Cov}_{\mu_t}(f, -\partial_t(W \circledast \nu_t)) \\ &= \mathrm{Cov}_{\mu_t}(f, -W \circledast (\nu_1 - \nu_0))\end{aligned}$$

and

$$|\nabla_x[W \circledast (\nu_1 - \nu_0)]| = |(\nabla_x W) \circledast (\nu_1 - \nu_0)| \le \|\nabla^2_{yx} W\|_\infty W_1(\nu_0, \nu_1).$$

Therefore using the Lipschitzian spectral gap estimate (38) in Lemma 12 for the generator $\mathcal{L}_{\nu_t}$,

$$\begin{aligned}\mathrm{Cov}_{\mu_t}(f, -W \circledast (\nu_1 - \nu_0)) &= \langle (-\mathcal{L}_{\nu_t})^{-1} f, \mathcal{L}_{\nu_t} W \circledast (\nu_1 - \nu_0)\rangle_{\mu_t} \\ &= \int \langle \nabla(-\mathcal{L}_{\nu_t})^{-1} f, \nabla W \circledast (\nu_1 - \nu_0)\rangle d\mu_t \\ &\le c_{Lip,m}\|\nabla^2_{xy} W\|_\infty W_1(\nu_0, \nu_1)\end{aligned}$$

Thus we have

$$\mu_1(f) - \mu_0(f) = \int_0^1 \frac{d}{dt}\mu_t(f)dt \le c_{Lip,m}\|\nabla^2_{xy} W\|_\infty W_1(\nu_0, \nu_1).$$

This means that $W_1(\Phi(\nu_0), \Phi(\nu_1)) \le c_{Lip,m}\|\nabla^2_{xy} W\|_\infty W_1(\nu_0, \nu_1)$ by Kantorovitch-Rubinstein's duality relation. The proof is so completed. □

Remark 22. Though $(M_1^2(\mathbb{R}^d), W_1)$ is not complete, the Banach's fixed point theorem works for the essential: let ν_∞ be the unique minimizer of E_f, then for any $\nu \in M_1^2(\mathbb{R}^d)$,

$$W_1(\Phi^n(\nu), \nu_\infty) \le [c_{Lip,m}\|\nabla_{xy} W\|_\infty]^n \cdot W_1(\nu, \nu_\infty), n \ge 0.$$

As for the mean field relative entropy, the Fisher-Donsker-Varadhan's information $I_W(\nu)$ can be also interpreted as the mean Fisher-Donsker-Varadhan's information per particle.

Lemma 23. **(convergence of the Fisher information)** *If $I(\nu|\alpha) < +\infty$,*

$$\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) \to I_W(\nu). \tag{49}$$

Proof. For every probability measure ν on $\mathbb{R}^d$ such that $I(\nu|\alpha) < +\infty$, by the Lyapunov function condition (H1) on V ([16]),

$$c_1 \int |x|^2 d\nu \le c_2 + I(\nu|\alpha) < +\infty.$$

As W has bounded second order derivatives, $\nabla_x W$ is of linear growth. Then $\nabla_x W \in L^2(\nu^{\otimes 2})$. By the law of large number for i.i.d. sequence, we have

$$\begin{aligned}
\frac{1}{N} I(\nu^{\otimes N}|\mu^{(N)}) &= \frac{1}{4N}\int |\nabla \log \frac{d\nu^{\otimes N}}{d\mu^{(N)}}|^2 d\nu^{\otimes N} \\
&= \frac{1}{4N}\int \sum_{i=1}^N |\nabla_{x_i} \log \frac{d\nu^{\otimes N}}{d\alpha^{\otimes N}} + \frac{1}{N-1}\sum_{j\neq i} \nabla_x W(x_i,x_j)|^2 d\nu^{\otimes N} \\
&= \int \frac{1}{4}|\nabla \log \frac{d\nu}{d\alpha}(x_1) + \frac{1}{N-1}\sum_{j=2}^N \nabla_x W(x_1,x_j)|^2 d\nu^{\otimes N} \\
&\to \frac{1}{4}\int |\nabla \log \frac{d\nu}{d\alpha}(x_1) + \int \nabla_x W(x_1,y)d\nu(y)|^2 d\nu(x_1) = I_W(\nu).
\end{aligned}$$

□

6.2. **Proof of Theorem 10. (1).** At first the minimizer ν_∞ of H_W is unique by Lemma 21.

(2). We may assume that $I(\nu|\alpha) < +\infty$, otherwise (34) is trivial for $I_W(\nu) = +\infty$. Since the Hessian $\nabla^2 V$ is lower bounded, and V satisfies the Lyapunov function condition (9), by Cattiaux-Guillin-Wu [12], α satisfies a log-Sobolev inequality. Then $H(\nu|\alpha) < +\infty$. By the log-Sobolev inequality of $\mu^{(N)}$ in Theorem 8,

$$\rho_{LS}(\mu^{(N)})H(\nu^{\otimes N}|\mu^{(N)}) \le 2I(\nu^{\otimes N}|\mu^{(N)})$$

and $\rho_{LS}(\mu^{(N)}) \ge \rho_{LS,m}/(1-\gamma_0)^2 > 0$. Dividing the two sides by N and letting N go to infinity, we get by Lemma 17 and Lemma 23,

$$\rho_{LS} H_W(\nu) \le 2I_W(\nu).$$

(3). By Otto-Villani [21] or Bobkov-Gentil-Ledoux [6], the log-Sobolev inequality implies the Talagrand's T_2 transportation inequality, i.e.

$$\rho_{LS}(\mu^{(N)})W_2^2(Q,\mu^{(N)}) \le 2H(Q|\mu^{(N)}),\ Q \in \mathcal{M}_1((\mathbb{R}^d)^N).$$

Applying it to $Q = \nu^{\otimes N}$ with $H(\nu|\alpha) < +\infty$, we obtain

$$\rho_{LS}(\mu^{(N)})\frac{1}{N}W_2^2(\nu^{\otimes N},\mu^{(N)}) \le \frac{1}{N}H(\nu^{\otimes N}|\mu^{(N)}).$$

Notice that

$$W_2^2(\nu^{\otimes N},\mu^{(N)}) \ge \sum_{i=1}^N W_2^2(\nu,\mu^{(N,i)}) = NW_2^2(\nu,\mu^{(N,1)})$$

where $\mu^{(N,i)}$ is the marginal distribution of x_i under $\mu^{(N)}$, which are all the same by the symmetry of $\mu^{(N)}$. Moreover by the uniqueness of ν_∞ and the large deviation principle of $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ under $\mu^{(N)}$ ([28]), for any $f \in C_b(\mathbb{R}^d)$,

$$\mu^{(N,1)}(f) = \int \frac{1}{N}\sum_{i=1}^N f(x_i)d\mu^{(N)} \to \nu_\infty(f),$$

i.e. $\mu^{(N,1)}$ converges weakly to ν_∞ . We obtain by Lemma 17 and the lower semi-continuity of W_2,

$$\rho_{LS} W_2^2(\nu,\nu_\infty) \le \rho_{LS} \liminf_{N\to\infty} W_2^2(\nu,\mu^{(N,1)}) \le 2H_W(\nu)$$

the desired Talagrand's type T_2-inequality for McKean-Vlasov equation.

(4). The exponential convergence in entropy (36) should be equivalent to the mean field log-Sobolev inequality (34) in part (2), basing on

$$-\frac{d}{dt}H_W(\nu_t) = 4I_W(\nu_t) \tag{50}$$

noted by Carrillo-McCann-Villani [10] in their convex framework. The proof of (50) demands the regularity of ν_t which requires the PDE theory of the McKean-Vlasov equation. That is why we prefer to give a rigorous probabilistic proof based directly on the log-Sobolev inequality of $\mu^{(N)}$ in Theorem 8.

For the exponential convergence (36), we may and will assume that $H_W(\nu_0) < +\infty$ and we fix the time $t > 0$. By Lemma 17,

$$\lim_{N\to\infty} \frac{1}{N} H(\nu_0^{\otimes N}|\mu^{(N)}) = H_W(\nu_0).$$

Moreover by the equivalence between the log-Sobolev inequality for $\mu^{(N)}$ and the exponential convergence in entropy of the law μ_t^N of $X_t^N = (X_t^{N,i})_{1\le i\le N}$ to $\mu^{(N)}$,

$$\begin{aligned} \frac{1}{N}H(\mu_t^N|\mu^{(N)}) &\le e^{-\rho_{LS}(\mu^{(N)})t/2}\frac{1}{N}H(\mu_0^N|\mu^{(N)}) \\ &= e^{-\rho_{LS}(\mu^{(N)})t/2}\frac{1}{N}H(\nu_0^{\otimes N}|\mu^{(N)}) < +\infty. \end{aligned} \tag{51}$$

Therefore $H(\mu_t^N|\alpha^{\otimes N}) < +\infty$ by Lemma 19. Since μ_t^N has finite second moment (easy from the SDE theory), and W has at most quadratic growth,

$$W(x_i, x_j) \in L^1(\mu_t^N).$$

From Lemma 18, we have

$$\frac{1}{N}H(\mu_t^N|\alpha^{\otimes N}) \ge H(\mu_t^{N,1}|\alpha).$$

And by the propagation of chaos (Lemma 20) and the lower semi-continuity of the relative entropy $\nu \to H(\nu|\alpha)$, $\liminf_{N\to\infty} H(\mu_t^{N,1}|\alpha) \ge H(\nu_t|\alpha)$.

So we get

$$\begin{aligned} \liminf_{N\to\infty}\frac{1}{N}H(\mu_t^N|\mu^{(N)}) &= \liminf_{N\to\infty}\left(\frac{1}{N}H(\mu_t^N|\alpha^{\otimes N}) + \int \frac{1}{N(N-1)}\sum_{1\le i<j\le N} W(x_i,x_j)d\mu_t^N + \frac{1}{N}\log \tilde{Z}_N\right) \\ &\ge H(\nu_t|\alpha) + \liminf_{N\to\infty}\frac{1}{2}\int W(x_1,x_2)d\mu_t^N - \inf_{\nu\in\mathcal{M}_1(\mathbb{R}^d)} E_f(\nu) \\ &= H(\nu_t|\alpha) + \frac{1}{2}\iint W(x_1,x_2)d\nu_t(x_1)d\nu_t(x_2) - \inf_{\nu\in\mathcal{M}_1(\mathbb{R}^d)} E_f(\nu) \\ &= H_W(\nu_t) \end{aligned}$$

by the W_2-propagation of chaos in Lemma 20. Plugging it into (51), we obtain the exponential convergence in entropy (36). That implies the W_2-exponential convergence (37) by Talagrand's type T_2-inequality (35). $\square$

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