Survey on the p-adic Hayman conjecture

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Introduction

We denote by $\mathbb{K}$ an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value $| \ . \ |$ and by $\mathbb{K}(x)$ the field of rational functions with coefficients in $\mathbb{K}$. Given $a \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(a, R)$ the disk $\{x \in \mathbb{K} \mid |x-a| \leq R\}$, by $d(a, R^-)$ the disk $\{x \in \mathbb{K} \mid |x-a| < R\}$ and by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |x-a| = r\}$. Next, we denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of analytic functions in $\mathbb{K}$ (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ (i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$). Similarly, we denote by $\mathcal{A}(d(a, R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(a, R^-)$ (i.e. the set of power series with a radius of convergence $\geq R$) [8], [11], [12], [13] and by $\mathcal{M}(d(a, R^-))$ its field of fractions and we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$-algebra of bounded analytic functions in $d(a, R^-)$ and by $\mathcal{M}_b(d(a, R^-))$ its field of fractions and we set $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ and $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Given $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(a, R^-))$), a value $b \in \mathbb{K}$ is called *an exceptional value for f* if $f - b$ has no zero in $\mathbb{K}$ (resp. in $d(a, R^-)$) and it is called *a quasi-exceptional value for f* if $f - b$ has finitely many zeros in $\mathbb{K}$ (resp. in $d(a, R^-)$).

In the complex field, in the fifties, Walter Hayman asked the question whether, given a meromorphic function g in the whole plane $\mathbb{C}$ and an integer $n \in \mathbb{N}$, the function $g'g^n$ might admit an exceptional value $b \neq 0$ or a quasi-exceptional value $b \neq 0$ [10]. W. Hayman showed that $g'g^n$ has no quasi-exceptional value, whenever $n \geq 3$. Next , the problem was solved for $n = 2$ by E. Mues in 1979 [15] and next, for $n = 1$, in 1995 by W. Bergweiler and A. Eremenko [1] and separately by H. Chen and M. Fang.

The same problem occurs on the field $\mathbb{K}$, both in $\mathcal{M}(\mathbb{K})$ and in a field $\mathcal{M}(d(a, R^-))$, $a \in \mathbb{K}$, $R > 0$. Several basic results will be necessary to examine this.

In ultrametric analysis as in complex analysis, we have this immediate correspondance:

Lemma 1: *Let $g \in \mathcal{M}(\mathbb{K})$ (resp. let $g \in \mathcal{M}(d(a, R^-))$, $a \in \mathbb{K}$, $R > 0$), let $f = \dfrac{1}{g}$ and let $n \in \mathbb{N}^*$. Then $g'g^n$ admits a quasi-exceptional value $b \in \mathbb{K}^*$ if and only if $f' + bf^{n+2}$ has finitely many zeros that are not zeros of f.*

Remark: We can also consider the same problem when $n = -1$ i.e. the question whether $f' + bf$ has infinitely many zeros. In $\mathbb{C}$ the well known counter-example furnished by the function $\exp(x)$ shows that $f' + f$ may have no zero. When $n = 0$, in $\mathbb{C}$ the well known counter-example furnished by the function $\tan(-x)$ shows that $f' + f^2$ may have no zero. On the field $\mathbb{K}$, we will examine the cases $n = -1$ and $n = 0$.

Henceforth, for $n \geq 3$, we will examine that problem by considering the set of zeros of $f' + bf^{n+2}$, with $b \neq 0$. In the field $\mathbb{K}$, two theorems are specific to p-adic analysis. Both are based on the following lemma. We set $m = n + 2$.

Notation: Let $f \in \mathcal{M}(d(0, R^-))$. For every $r \in]0, R[$, $|f(x)|$ has a limit when $|x|$ tends to r while staing different from r and that limit is denoted by $|f|(r)$ [8].

Let $\mu = \log r$. We set $\Psi(f, \mu) = \log(|f|(r))$ We denote by $\nu^+(f, \mu)$ the difference between the number of zeros and the number of poles of f in $d(0, r)$ and we denote by $\nu^-(f, \mu)$ the difference between the number of zeros and the number of poles of f in $d(0, r^-)$.

The following Lemma 1 and 2 are classical [17]:

Lemma 2: *Let $f \in \mathcal{M}(\mathbb{K})$. Then $\nu^+(f, \mu)$ is the right side derivative of $\Psi(f, \mu)$ and $\nu^+(f, \mu)$ is the left side derivative of $\Psi(f, \mu)$.*

Lemma 3: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $f \in \mathcal{M}(d(a, R^-))$, $a \in \mathbb{K}$, $R > 0$), suppose that f admits infinitely many zeros and suppose that there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n\to+\infty} r'_n = +\infty$ (resp. $\lim_{n\to+\infty} r'_n = \lim_{n\to+\infty} r''_n = R$) and such that $|(f' + f^m)|(r) = |f^m|(r)$ $\forall r \in \bigcup_{n\in\mathbb{N}} [r'_n, r''_n]$. Let $m \in \mathbb{N}^*$ be $\neq 2$. Then $f' + f^m$ has infinitely many zeros that are not zeros of f.*

Proof: Let $J = \bigcup_{n\in\mathbb{N}} [r'_n, r''_n]$. When r is big enough, we have $|f^m|(r) > |f'|(r)$ therefore

$$(1) \quad \nu^+(f' + f^m, \log r) = \nu^+(f^m, \log r),\ \nu^-(f' + f^m, \log r) = \nu^-(f^m, \log r)\ \forall r \in J.$$

Consequently, in each disk $d(0, r)$ with $r \in J$, f and $f' + f^m$ have the same difference between the number of zeros and poles. Now, if $m \geq 3$ the poles of $f' + f^m$ and f^m are the same taking multiplicity into account. And when $m = 1$, each pole of f is a pole of $f' + f$ with a greater order. Consequently, for each $r \in J$, the number of zeros of $f' + f^m$ in $d(0, r)$ is superior or equal to this of f^m.

Now, for each $n \in \mathbb{N}$, let s_n be the number of distinct zeros of f in $d(0, r''_n)$. Since f has infinitely many zeros, the sequence s_n is increasing and tends to $+\infty$. On the other hand, for each zero α of order u of f, either α is not a zero of $f' + f^m$ (when $u = 1$), or it is a zero of order $u - 1$. Consequently, the number of zeros of $f' + f^m$ in $d(0, r''_n)$ which are not zeros of f is at least s_n. Thus we have proved that $f' + f^m$ has infinitely many zeros that are not zeros of f.

Remark: Relation (1) above does not hold when $m = 2$ because poles of f^2 and f' may have the same order and therefore may kill each other.

In most of results, we will use the ultrametric Nevanlinna theory [33], [35]. The Nevanlinna Theory was made by Rolf Nevanlinna on complex functions [16], [10]. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f - b,\ b \in \mathbb{C}$.

A similar theory for functions in a p-adic field was constructed by A. Boutabaa [6].

Notations: Given three functions $\phi,\ \psi,\ \zeta$ defined in an interval $J =]a, +\infty[$ (resp. $J =]a, R[$), with values in $[0, +\infty[$, we shall write $\phi(r) \leq \psi(r) + O(\zeta(r))$ if there exists a constant $b \in \mathbb{R}$ such that $\phi(r) \leq \psi(r) + b\zeta(r)$. We shall write $\phi(r) = \psi(r) + O(\zeta(r))$ if $|\psi(r) - \phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r) \leq \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to $\mathbb{R}$ such that $\lim_{r \to +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \to R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) \leq \psi(r) + h(r)$. And we shall write $\phi(r) = \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to $\mathbb{R}$ such that $\lim_{r \to +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \to R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) = \psi(r) + h(r)$.

The p-adic Nevanlinna Theory was first stated and correctly proved by A. Boutabaa in $\mathcal{M}(\mathbb{K})$ [6]. In [7] the theory was extended to functions in $\mathcal{M}(d(0, R^-))$ by taking into account Lazard's problem [14].

Throughout the next paragraphs, we will denote by I the interval $[t, +\infty[$ and by J an interval of the form $[t, R[$ with $t > 0$.

We have to introduce the counting function of zeros and poles of f, counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way:

Let $\sigma(r)$ be the number of distinct zeros of f in $d(0, r)$ and let Let $(a_n),\ 1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f in $d(0, r)$, of respective order s_n.

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n(\log r - \log |a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity.*

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$\overline{Z}(r, f) = \overline{\omega_0}(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$ and so, $\overline{Z}(r, f)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity.*

In the same way, we denote by $\tau(r)$ the number of distinct poles of f in $d(0,r)$ and then, considering the finite sequence (b_n), $1 \leq n \leq \tau(r)$ of poles of f in $d(0,r)$, with respective multiplicity order t_n, we put

$N(r,f) = \max(-\omega_0(f),0)\log r + \sum_{n=1}^{\tau(r)} t_n(\log r - \log|b_n|)$ and then $N(r,f)$ is called *the counting function of the poles of f, counting multiplicity*

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega_0}}(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega_0}}(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$\overline{N}(r,f) = \overline{\overline{\omega_0}}(f)\log r + \sum_{n=1}^{\tau(r)} (\log r - \log|b_n|)$ and then $\overline{N}(r,f)$ is called *the counting function of the poles of f, ignoring multiplicity*

Now we can define the Nevanlinna function $T(r,f)$ in I or J as
$T(r,f) = \max(Z(r,f), N(r,f))$ and the function $T(r,f)$ is called *characteristic function of f or Nevanlinna of f.*

Finally, if S is a subset of $\mathbb{K}$ we will denote by $Z_0^S(r,f')$ the counting function of zeros of f', excluding those which are zeros of $f-a$ for any $a \in S$.

Remark: If we change the origin, the functions Z, N, T are not changed, up to an additive constant.

In a p-adic field such as $\mathbb{K}$, the first Main Theorem is almost immediate:

Theorem A: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0,R^-))$) have no zero and no pole at 0. Then $\log(|f|(r)) = \Psi(f,\log r) = \log(|f(0)|) + Z(r,f) - N(r,f)$.*

Corollary A.1: *Let f, $g \in \mathcal{M}(\mathbb{K})$ (resp. f, $g \in \mathcal{M}(d(0,R^-))$). Then $Z(r,fg) \leq Z(r,f) + Z(r,g)$, $N(r,fg) \leq N(r,f) + N(r,g)$, $T(r,fg) \leq T(r,f) + T(r,g)$, $T(r,f+g) \leq T(r,f) + T(r,g) + O(1)$, $T(r,cf) = T(r,f)$ $\forall c \in \mathbb{K}^*$, $T(r,\frac{1}{f}) = T(r,f)$).*

If $f,g \in \mathcal{A}(\mathbb{K})$ (resp. if f, $g \in \mathcal{A}(d(0,R^-))$), then $Z(r,fg) = Z(r,f)+Z(r,g)$, $T(r,f) = Z(r,f))$, $T(r,fg) = T(r,f) + T(r,g) + O(1)$ and $T(r,f+g) \leq \max(T(r,f),T(r,g))$.

We can now state the famous p-adic Second Main Theorem:

Theorem B: *Let $\alpha_1,...,\alpha_q \in \mathbb{K}$, with $q \geq 2$, let $S = \{\alpha_1,...,\alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0,R^-))$). Then*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r,f-\alpha_j) + \overline{N}(r,f) - Z_0^S(r,f') - \log r + O(1) \quad \forall r \in I \ \ (\textit{resp. } \forall r \in J).$$

Definitions and notation: For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a,R^-))$) we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a,R^-))$) the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(d(a,R^-))$) such that $T(r,h) = o(T(r,f))$ when r tends to $+\infty$ (resp. when r tends to R). Similarly, if $f\in\mathcal{A}(\mathbb{K})$ (resp. $f\in\mathcal{A}(d(a,R^-))$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a,R^-))$) the set $\mathcal{M}_f(\mathbb{K})\cap\mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a,R^-))\cap\mathcal{A}(d(a,R^-))$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(a,R^-))$) are called *small functions with respect to* f. Similarly, if $f\in\mathcal{A}(\mathbb{K})$ (resp. $f\in\mathcal{A}(d(a,R^-))$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a,R^-))$) are called *small functions with respect to* f.

According to classical results [11], [18], we have the following Theorem C:

Theorem C: *Let $f\in\mathcal{A}(\mathbb{K})$ (resp. $f\in\mathcal{A}_u(d(0,R^-))$) and let $w_1,\ w_2\in\mathcal{A}_f(\mathbb{K})$ (resp. $w_1,w_2\in\mathcal{A}_f(d(0,R^-))$) be distinct. Then $T(r,f)\leq\overline{Z}(r,f-w_1)+\overline{Z}(r,f-w_2)+o(T(r,f))$.*

Definitions and notation: Given $f,\ g\in\mathcal{M}(d(0,R^-))$, we denote by $W(f,g)$ the Wronskian of f and g i.e. $f'g-fg'$.

In [5], the following results are proven:

Theorem D: *Let $f,\ g\in\mathcal{A}(\mathbb{K})$ be such that $W(f,g)$ is a non-identically zero polynomial. Then both $f,\ g$ are polynomials.*

Notation: Let $f\in\mathcal{A}(\mathbb{K})$. We can factorize f in the form $\overline{f}\widetilde{f}$ where the zeros of $\overline{f}$ are the distinct zeros of f each with order 1. Moreover, if $f(0)\neq 0$ we will take $\overline{f}$ such that $\overline{f}(0)=1$.

Theorem E: *Let $f\in\mathcal{M}(\mathbb{K})$ have finitely many multiple poles, such that for certain $b\in\mathbb{K}$, $f'-b$ has finitely many zeros. Then f belongs to $\mathbb{K}(x)$.*

Notation: Let $f\in\mathcal{M}(d(0,R^-))$. For each $r\in]0,R[$, we denote by $\zeta(r,f)$ the number of zeros of f in $d(0,r)$, taking multiplicity into account and set $\tau(r,f)=\zeta(r,\frac{1}{f})$. Similarly, we denote by $\beta(r,f)$ the number of multiple zeros of f in $d(0,r)$, each counted with its multiplicity and we set $\gamma(r,f)=\beta(r,\frac{1}{f})$.

Theorem F: *Let $f\in\mathcal{M}(\mathbb{K})$ be such that for some $c,d\in]0,+\infty[$, $\gamma(r,f)$ satisfies $\gamma(r,f)\leq cr^d$ in $[1,+\infty[$. If f' has finitely many zeros, then $f\in\mathbb{K}(x)$.*

Corollary F.1: *Let f be a meromorphic function on $\mathbb{K}$ such that, for some $c,d\in]0,+\infty[$, $\gamma(r,f)$ satisfies $\gamma(r,f)\leq cr^d$ in $[1,+\infty[$. If for some $b\in\mathbb{K}$ $f'-b$ has finitely many zeros, then f is a rational function.*

First results [17]

We will now prove together the following Theorems 1 and 2.

Theorem 1: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ satisfy $\limsup_{r \to \infty} |f|(r) > 0$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3. Then $f' + bf^m$ has infinitely many zeros that are not zeros of f.*

Theorem 2: *Let $f \in \mathcal{M}_u(d(a, R^-))$ satisfy $\limsup_{r \to R} |f|(r) = +\infty$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3. Then $f' + bf^m$ has infinitely many zeros that are not zeros of f.*

Proof : Without loss of generality, we can assume $b = 1$ and when $f \in \mathcal{M}(d(a, R^-))$, we may assume $a = 0$. By hypotheses, there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n \to +\infty} r'_n = +\infty$ (resp. $\lim_{n \to +\infty} r'_n = \lim_{n \to +\infty} r''_n = R$) and such that, putting $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$, we have $\liminf_{\substack{r \to \infty, \\ r \in J}} |f|(r) > 0$ (resp. $\lim_{\substack{r \to R^- \\ r \in J}} |f|(r) = +\infty$).

Suppose first we assume the hypothesis of Theorem 1. Let $M = \dfrac{\liminf_{r \to +\infty} |f|(r)}{2}$. We will prove that there exists $t > 0$ such that $|f' + f^m|(r) = |f^m|(r)\ \forall r \in J \cap [t, +\infty[$. We know that $|f'|(r) \leq \dfrac{|f|(r)}{r}$. Consequently, when r lies in J, there exists $s > 0$ such that $|f|(r) \geq M\ \forall r \in [s, +\infty[\cap J$.

$$\big(|f|(r)\big)^m \geq |f|(r) M^{m-1} \geq r|f'|(r) M^{m-1}.$$

Next, when r is big enough, rM^{m-1} is greater than 1, hence $(|f|(r))^m > |f'|(r)$. Thus there exists $t \geq s$ such that $(|f|(r))^m > |f'|(r)\ \forall r \in J \cap [t, +\infty[$. Let $J' = J \cap [t, +\infty[$. And hence we have $|f' + f^m|(r) = |f^m|(r)\ \forall r \in J'$.

Suppose now that we assume the hypothesis of Theorem 2 We have
$|f'|(r) \leq \dfrac{|f|(r)}{r} \leq \dfrac{|f|(r)}{R}$. Set $B = \dfrac{1}{R}$. Then we have

$$\big(|f|(r)\big)^m \geq B|f'|(r)(|f|(r))^{m-1}.$$

Now, when r is close enough to R, $r \in J$, $B|f(x)|^{m-1}$ is strictly greater than 1, hence $(|f|(r))^m > |f'|(r)$. Thus there exists $t > 0$ such that $(|f|(r))^m > |f'|(r)\ \forall r \in [t, +\infty[\cap J$. We can set again $J' = J \cap [t, R[$ and then we have $|f' + f^m|(r) = |f^m|(r)\ \forall r \in J'$

We can now conclude in both theorems 1 and 2. For each $n \in \mathbb{N}$, let q_n be the number of zeros of f in $d(0, r''_n)$. Suppose the sequence $(q_n)_{n \in \mathbb{N}}$ is bounded. Then, f has finitely many zeros, hence it is of the form $\dfrac{P}{h}$ with $P \in \mathbb{K}[x]$ and $h \in \mathcal{A}_u(d(0, R))$. Consequently, we have $\lim_{r \to +\infty} |f|(r) = 0$, a contradiction to the hypothesis in both theorems. Therefore, the sequence $(q_n)_{n \in \mathbb{N}}$ which is increasing by definition, tends to $+\infty$. Now, in each Theorems 1 and 2 we may apply Lemma 3 showing that $f' + f^m$ has infinitely many zeros that are not zeros of f.

Consider now the case $m = 1$. We can have a better conclusion in $\mathcal{M}(\mathbb{K})$.

Theorem 3: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. For each $b \in \mathbb{K}^*$, $f' + bf$ has infinitely many zeros that are not zeros of f.*

Proof: Without loss of generality, we can assume again $b = 1$. We have $|f'|(r) < |f|(r)$ when r is big enough and hence $|f' + f|(r) = |f|(r)$ in an interval $I = [s, +\infty[$. Suppose first that f has infinitely many zeros. We can then apply Lemma 3 and get the conclusion.

Suppose now that f has finitely many zeros. Then f has infinitely many poles c_n of respective order t_n. Since $\mathbb{K}$ has characteristic zero, f' admits each c_n as a pole of order $t_n + 1$ and similarly, $f' + f$ also admits each c_n as a pole of order $t_n + 1$. Thus, we have $N(r, f' + f) = N(r, f) + \overline{N}(r, f)$. But since $|f' + f|(r) = |f|(r)$ holds in I, we have $\Psi(f' + f, \log r) = \Psi(f, \log r)\ \forall r \in I$ hence, by Lemma 2, $\nu(f' + f, \log r) = \nu(f, \log r)\ \forall r \in I$ and hence $Z(r, f' + f) - N(r, f' + f) = Z(r, f) - N(r, f)$, therefore $Z(r, f' + f) - (N(r, f) + \overline{N}(r, f)) = Z(r, f) - N(r, f)$ and hence $Z(r, f' + f) = Z(r, f) + \overline{N}(r, f)$. Since we have supposed that f has finitely many zeros and since f has infinitely many poles, $f' + f$ has infinitely many zeros and all but finitely many are not zeros of f.

Theorem 4: *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $a \in \mathbb{K}$, let $R \in]0, +\infty[$ and let $f \in \mathcal{A}_u(da(, R^)))$. For each $b \in \mathbb{K}^*$, $f' + bf^2$ has infinitely many zeros that are not zeros of f.*

Proof: Without loss of generality, we can assume $b = 1$ and $a = 0$. Clearly, when r is big enough, in $]0, +\infty[$ (resp. in $]0, R[$), we have $|f' + f^2|(r) = |f^2|(r)$ therefore f^2 and $f' + f^2$ have the same number of zeros in $C(0, r)$. Let $\alpha \in C(0, r)$ be a zero of f of order q. When r is big enough, it is a zero of order $2q$ for f^2 and it is a zero of order $q - 1$ for $f' + f^2$. Consequently, $f' + f^2$ has at least $q + 1$ zero in $C(0, r)$ that are not zeros of f (taking multiplicity into account). This is true for every such zeros of f an hence $f' + f^2$ has infinitely many zeros that are not zeros of f.

Corollary 4.1: *Let $m \in \mathbb{N}$ be ≥ 1, let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}(x)$. For each $b \in \mathbb{K}^*$, $f' + bf^m$ has infinitely many zeros that are not zeros of f.*

Corollary 4.2: *Let $m \in \mathbb{N}$ be ≥ 2, let $a \in \mathbb{K}$, let $R \in]0, +\infty[$ and let $f \in \mathcal{A}_u(da(, R^-)))$. For each $b \in \mathbb{K}^*$, $f' + bf^m$ has infinitely many zeros that are not zeros of f.*

Theorem 5 is given in [4]:

Theorem 5: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $a \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$ and let $f \in \mathcal{M}(d(a, R^-)))$ and let $m \in \mathbb{N}$. If $m \geq 5$ then for each $b \in \mathbb{K}^*$, $f' + bf^m$ has infinitely many zeros that are not zeros of f.*

If $m = 4$, if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ and if f admits at least s multiple zeros and at least t multiple poles, then $f' + bf^4$ admits a number of zeros that are not zeros of f (taken account of multiplicity) which is strictly superior to $\dfrac{s+t}{2}$.

Proof: We know that the zeros of $f' + bf^m$ in $\mathbb{K}$ are the same as in a spherically complete algebraically closed extension $\widehat{\mathbb{K}}$ of $\mathbb{K}$. So, for simplicity, we can suppose that the field $\mathbb{K}$

is spherically complete without loss of generality. We can also suppose that $b = 1$. Then if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ we can obviously we can write $f = \frac{h}{l}$ with $h,\ l \in \mathcal{A}(\mathbb{K})$, having no common zeros and if $f \in \mathcal{M}(d(a, R^-))$, since $\mathbb{K}$ is spherically complete, we can write $f = \frac{h}{l}$ with $h,\ l \in \mathcal{A}(d(a, R^-))$, having no common zeros again.

Let $g = \frac{1}{f}$ and let $n = m - 2$. So, by Lemma 1, the problem is reduced to show that $g'g^n - 1$ has infinitely many zeros. Then, $g'g^n - 1 = \frac{(l'h - hl')l^n - h^{n+2}}{h^{n+2}}$ and since $h,\ l$ have no common bzeros, this is of the form $\frac{P}{h^{n+2}}$ where P is a polynomial of degree q. Now, set $F = (l'h - hl')l^n$. Applying Theorem C to F we have

(1) $T(r, F) = Z(r, F) + O(1) \le \overline{Z}(r, F) + \overline{Z}(r, F - P) + T(r, P) + O(1)$. By (1) we derive $Z(r, l'h - h'l) + nZ(r, l) \le \overline{Z}(r, l'h - h'l) + \overline{Z}(r, l) + \overline{Z}(r, F - P) + T(r, P) + O(1)$. Actually, $\overline{Z}(r, F - P) = \overline{Z}(r, h)$, hence $nZ(r, l) \le \overline{Z}(r, l) + \overline{Z}(r, h) + T(r, P) + O(1)$ and hence $(n - 1)Z(r, l) \le Z(r, h) + T(P) + O(1)$. But since $T(r, P) = q \log r + O(1)$, we have

(2) $(n - 1)Z(r, l) \le Z(r, h) + q \log r + O(1)$

Now, consider the hypothesis $f \in \mathcal{M}(\mathbb{K})$. By Theorem 1, if $\liminf_{r\to+\infty} |f|(r) > 0$ i.e. if $\liminf_{r\to+\infty} Z(r, f) - N(r, f) > -\infty$ the claim is proved. Consequently, if the claim is not true, we can assume

(3) $\liminf_{r\to+\infty} Z(r, f) - N(r, f) = -\infty$

But we see that (3) is impossible whenever $n \ge 3$, i.e. $m \ge 5$.

Now, suppose $m = 4$ .i.e $n = 2$. More precisely $\overline{Z}(r, l) \le Z(r, l) - \frac{s \log r}{2}$ and $\overline{Z}(r, h) \le Z(r, h) - \frac{t \log r}{2}$, so by Relation (1) we have

(4) $(n - 1)Z(r, l) \le Z(r, h) + (q - \frac{s + t}{2}) \log r + O(1)$.

Then Relation (3) implies $q - \frac{s + t}{2} > 0$ and hence $f'f^n$ admits a number of zeros strictly superior to $\frac{s + t}{2}$.

Now, suppose that $f \in \mathcal{M}(d(0, R^-))$. By Theorem 2, if $\lim_{r\to R^-} |f|(r) = +\infty$ i.e. if $\liminf_{r\to R^-} Z(r, f) - N(r, f) = +\infty$ the claim is proved. Consequently, if the claim is not true, we can assume

(5) $\liminf_{r\to R^-} Z(r, f) - N(r, f) < +\infty$.

But by (2), we see that (5) is impossible whenever $n \ge 3$ i.e. $m \ge 5$.

Corollary 4.1: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. Then for each $n \ge 3$ $f'f^n$ has infinitely many zeros that are not zeros of f.*

Corollary 4.2: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have s multiple zeros and t multiple poles. Let $b \in \mathbb{K}^*$. Then $f' + bf^4$ has at least $s+t+1$ zeros taking account of multiplicity. Particularly, if f has infinitely many multiple zeros or poles, then $f' + bf^4$ has infinitely many zeros that are not zeros of f.*

Corollary 4.3: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have s multiple zeros and t multiple poles. Given $b \in \mathbb{K}^*$, If f has infinitely many multiple zeros or poles, then $f'f^2 - b$ has infinitely many zeros.*

Case $n = 2, \ m = 4$

We will now thorougly examine the situation when $m = 4$ i.e. $n = 2$, as made in [9]. This requires several basic lemmas.

Lemma 4: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and such that f' has finitely many multiple zeros. Then $\frac{f''f}{(f')^2}$ has no quasi-exceptional value.*

Proof: Let $g = \frac{f}{f'}$. A pole of g is a zero of f', hence by hypothesis, g has finitely many multiple poles. Consequently, by Theorem E, g' has no quasi-exceptional value. And hence neither has $1 - g'$. But $g' = \frac{(f')^2 - f''f}{(f')^2} = 1 - \frac{f''f}{(f')^2}$. Therefore $\frac{f''f}{(f')^2}$ has no quasi-exceptional value.

Lemma 5: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many multiple zeros. Then $f''f + 2(f')^2$ has infinitely many zeros that are not zeros of f.*

Proof: Suppose first that f' has infinitely many multiple zeros. Since f has finitely many multiple zeros, the zeros of f' are not zeros of f except at most finitely many. Hence f' has infinitely many multiple zeros that are not zeros of f. And then, they are zeros of f'', hence of $f''f + 2(f')^2$, which proves the statement.

So we are now led to assume that f' has finitely many multiple zeros. By Lemma 4 $\frac{f''f + 2(f')^2}{(f')^2}$ has infinitely many zeros. Let $c \in \mathbb{K}$ be a pole of order q of f. Without loss of generality, we can suppose $c = 0$. The beginning of the Laurent developpement of f at 0 is of the form $\frac{a_{-q}}{x^q} + \frac{\varphi(x)}{x^{q-1}}$ whereas $\varphi \in \mathcal{M}(\mathbb{K})$ has no pole at 0. Consequently, $\frac{f''f + 2(f')^2}{(f')^2}$ is of the form

$$\frac{(a_{-q})^2(3q^2 + q) + x\phi(x)}{(a_{-q})^2(q^2) + x\psi(x)}$$

whereas $\phi, \ \psi \in \mathcal{M}(\mathbb{K})$ have no pole at 0. So, the function $\frac{f''f + 2(f')^2}{(f')^2}$ has no zero at 0.

Therefore, each zero of $\dfrac{f''f+2(f')^2}{(f')^2}$ is a zero of $f''f+2(f')^2$ and hence $f''f+2(f')^2$ has infinitely many zeros.

Now, let us show that the zeros of $f''f+2(f')^2$ are not zeros of f, except maybe finitely many. Let c be a zero of $f''f+2(f')^2$ and suppose that c is a zero of f. Then, it is a zero of f' and hence it is a multiple zero of f. But by hypotheses, f has finitely many multiple zeros, hence the zeros of $f''f+2(f')^2$ are not zeros of f, except at most finitely many. That finishes proving the claim.

Lemma 6: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let $b \in \mathbb{K}^*$ be such that $f^2f'-b$ has finitely many zeros. Then, $N(r,f) \leq Z(r,f)+O(1)$.*

Proof: Let $F=f^2f'$. Since $F-b$ is transcendental and has finitely many zeros, it is of the form $\dfrac{P(x)}{h(x)}$ with $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$. Consequently, $|F|(r)$ is a constant when r is big enough and therefore, by Theorem A we have $Z(r,F)=N(r,F)+O(1)$ when r is big enough. Now, $Z(r,F)=2Z(r,f)+Z(r,f')$ and, by Theorem A $Z(r,f') \leq Z(r,f)+\overline{N}(r,f)-\log r+O(1)$. On the other hand, by Theorem A again, we have $N(r,F)=3N(r,f)+\overline{N}(r,f)$. Consequently, $3N(r,f)+\overline{N}(r,f) \leq 3Z(r,f)+\overline{N}(r,f)-\log r+O(1)$, which proves the claim.

Theorem 6: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ Then for each $b \in \mathbb{K}^*$, $f'f^2-b$ has infinitely many zeros.*

Proof: Let $b \in \mathbb{K}$ and suppose that the claim is wrong, i.e. $f^2f'-b$ has q zeros, taking multiplicity into account. By Theorem 5, we may assume that f has finitely many multiple zeros and finitely multiple poles. Set $F=f^2f'$. Then $F'=f(f''f+2(f')^2)$. By Lemma 5, $f''f+2(f')^2$ has infinitely many zeros that are not zeros of f. Consequently, F' admits for zeros: the zeros of f and the zeros of $f''f+2(f')^2$. And by Lemma 4, there exists a sequence of zeros of $f''f+2(f')^2$ that are not zeros of f.

Let $S=\{0,b\}$ and let $Z_0^S(r,F')$ be the counting function of zeros of F' when $F(x)$ is different from 0 and b. Since $F-b$ has finitely many zeros, the zeros c of F' which are not zeros of f cannot satisfy $F(c)=b$ except at most finitely many. Consequently, there are infinitely many zeros of F' counted by the counting function $Z_0^S(r,F')$ and hence for every fixed integer $t \in \mathbb{N}$, we have

(1) $$Z_0^S(r,F') \geq t\log r+O(1).$$

Let us apply Theorem B to F. We have

(2) $$T(r,F) \leq \overline{Z}(r,F)+\overline{Z}(r,F-b)+\overline{N}(r,F)-Z_0^S(r,F')-\log(r)+O(1).$$

Now, we have

(3) $$\overline{Z}(r,F) \leq Z(r,f)+Z(r,f')$$

(4) $$\overline{N}(r,F)=\overline{N}(r,f)$$

and since the number of zeros of $F-b$ is q, taking multiplicity into account,

(5) $$\overline{Z}(r,F-b)\leq q\log r+O(1).$$

Consequently, by (2), (3), (4), (5) we obtain

(6) $$T(r,F)\leq Z(r,f)+Z(r,f')+\overline{N}(r,f)-Z_0^S(r,F')+(q-1)\log r+O(1).$$

On the other hand, by construction, $T(r,F)\geq Z(r,F)=2Z(r,f)+Z(r,f')$ hence by (6) we obtain (7):

(7) $$Z(r,f)\leq \overline{N}(r,f)-Z_0^S(r,F')+(q-1)\log r+O(1).$$

Now, by Lemma 6, we have $N(r,f)\leq Z(r,f)+O(1)$ hence by (7) we obtain $0\leq (q-1)\log r-Z_0^S(r,F')+O(1)$ and hence by (1), fixing $t>q-1$ we can derive $0\leq (q-1)\log r-t\log r+O(1)$, a contradiction. That finishes the proof of Theorem 6.

By Lemma 1, Theorems 5 and 6 we can now state the general result on the p-adic Hayman conjecture:

Corollary 6.1: *Let $f\in\mathcal{M}(\mathbb{K})$ be transcendental. Then for every $n\in\mathbb{N}$, $n\geq 2$, for every $b\in\mathbb{K}^*$, $f^2f'-b$ has infinitely many zeros and for every $m\geq 4$, $f'+bf^m$ has infinitely many zeros that are not zeros of f.*

Case $n=1,\ m=3$

Concerning the case $m=3$ i.e. $n=1$ which remains unsolved, Corollary 6.1 has an immediate application to the conjecture with additional hypotheses [2].

Theorem 7: *Let $f\in\mathcal{M}(\mathbb{K})$. Suppose that there exists $c,d\in]0,+\infty[$, such that $\tau(r,f)\leq cr^d$ $\forall r\in[1,+\infty[$. If $f'f^n-b$ has has finitely many zeros for some $b\in\mathbb{K}$, with $n\in\mathbb{N}$, then $f\in\mathbb{K}(x)$.*

Proof: Suppose f is transcendental. By hypothesis, f^{n+1} satisfies $\zeta(r,\frac{1}{f^{n+1}})=\tau(r,f^{n+1})\leq c(n+1)r^d$ $\forall r\in[1,+\infty[$ hence by Corollary 6.1, $f'f^n$ has no quasi-exceptional value.

Theorem 7 may be writen in another way:

Corollary 7.1: *Let $f\in\mathcal{M}(\mathbb{K})\setminus\mathbb{K}(x)$. Suppose that there exists $c,d\in]0,+\infty[$, such that $\zeta(r,f)\leq cr^d$ $\forall r\in[1,+\infty[$. Then for all $m\in\mathbb{N}$, $m\geq 3$ and for all $b\in\mathbb{K}$, $f'-bf^m$ admits infinitely many zeros that are not zeros of f.*

Proof: We set $g = \dfrac{1}{f}$. Then by Theorem 7 $g'g^{m-2}$ has no quasi-exceptional value. Consequently, given $b \in \mathbb{K}^*$, $g'g^{m-2} + b$ has infinitely many zeros and hence $f' - bf^m$ has infinitely many zeros that are not zeros of f. Next, if $b = 0$, by Theorem F, f' has infinitely many zeros.

Theorem 8: *Let $f \in \mathcal{M}(\mathbb{K})$. Suppose that there exists $c, d \in]0, +\infty[$, such that $\beta(r, f) \leq cr^d$ $\forall r \in [1, +\infty[$. Then, for all $b \in \mathbb{K}$, $\dfrac{f'}{f^2} - b$ has infinitely many zeros.*

Proof: Set $g = \dfrac{1}{f}$ again. Since the poles of g are the zeros of f, we have $\gamma(r, g) \leq cr^d$. Consequently, by Corollary F.1, g' has no quasi-exceptional value.

Remark: Using Theorem 8 to study the zeros of $f' - bf^2$ that are not zeros of f is not so immediate, as we will see below because of residues of f at poles of order 1. Of course, if $\frac{1}{f}$ is an affine function, $f' + f^2$ has no zeros, except if it is identically zero. And if it is not identically zero, the residue at the pole is not 1 in the general case.

Case $n = 0$ **i.e.** $m = 2$

As we noticed at the beginning, due to the counter-example provided by the function tan, the case $n = 0$ has no solution in $\mathbb{C}$. However, we can notice certain conclusions.

Lemma 7: *Let $f = \dfrac{h}{l} \in \mathcal{M}(\mathbb{K})$ with $h,\ l \in \mathcal{A}(\mathbb{K})$ having no common zero, let $b \in \mathbb{K}^*$ and let $a \in \mathbb{K}$ be a zero of $h'l - hl' + bh^2$ that is not a zero of $f' + bf^2$. Then a is a pole of order 1 of f and $\mathrm{res}(f, a) = \dfrac{1}{b}$.*

Proof: Clearly, if $l(a) \neq 0$, a is a zero of $f' + bf^2$. Hence, a zero a of $h'l - hl' + bh^2$ that is not a zero of $f' + bf^2$ is a pole of f. Now, when $l(a) = 0$, we have $h(a) \neq 0$ hence $l'(a) = bh(a) \neq 0$ and therefore a is a pole of order 1 of f such that $\dfrac{h(a)}{l'(a)} = \dfrac{1}{b}$. But since a is a pole of order 1, we have $\mathrm{res}(f, a) = \dfrac{h(a)}{l'(a)}$, which ends the proof.

Theorem 9 is not a result specific to p-adic analysis but it will be useful in Theorem 10.

Theorem 9: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $a \in \mathbb{K}$, let $f \in \mathcal{M}(d(a, R^-))$), let $b \in \mathbb{K}^*$ and let $\alpha \in \mathbb{K}$ (resp. let $\alpha \in d(a, R^-)$) be a point that is not a zero of f and such that the residue of f at α is different from $\dfrac{1}{b}$. Then α is a zero of $f' + bf^2$ if and only if it is a zero of $\dfrac{f'}{f^2} + b$. Moreover, if it is a zero of both functions, it has the same multiplicity with both.*

Proof: Suppose first α is a zero of $f' + bf^2$. If α is not a pole of f, of course it is a zero of $\frac{f'}{f^2} + b$ with same multiplicity. Suppose now that α is a pole of f: since it is not a pole of $f' + bf^2$ it must be a pole of order 1 of f. Without loss of generality, we may assume that $\alpha = 0$ (resp. $a = \alpha = 0$). Consider the Laurent series of f at 0: $f(x) = \frac{a_{-1}}{x} + a_0 + a_1 x + x^2\phi(x)$ with $\phi \in \mathcal{M}(\mathbb{K})$ (resp. $\phi \in \mathcal{M}(d(0, R^-))$ and $\phi(0) \neq \infty$. Then $f' + bf^2$ is of the form

$$f'(x) + bf(x)^2 = \frac{a_{-1}(-1 + ba_{-1})}{x^2} + \frac{2ba_0a_1}{x} + a_1 + b(a_0^2 + 2a_1a_{-1}) + x\xi(x)$$

with $\xi \in \mathcal{M}(\mathbb{K})$ (resp. $\xi \in \mathcal{M}(d(0, R^-))$ and $\xi(0) \neq \infty$ and hence, we have $a_{-1}(-1 + ba_{-1}) = 0$, $a_0a_{-1} = 0$, $a_0^2 + 2a_1a_{-1} = 0$. Since by hypothesis $\text{res}(f, \alpha) \neq -\frac{1}{b}$ we have $(1 + ba_{-1}) \neq 0$, hence $a_{-1} = 0$, a contradiction. Consequently, every zero of $f' + bf^2$ that is not a zero of f is a zero of $\frac{f'}{f^2} + b$ with same multiplicity.

Conversely, suppose now that α is a zero of $\frac{f'}{f^2} + b$. If α is not a pole of f, it is a zero of $f' + bf^2$, with the same multiplicity, because by hypothesis it is not a zero of f. Now suppose that α is a zero of $\frac{f'}{f^2} + b$ and is a pole of f. Clearly, it is a pole of order 1 and again, we may assume that $\alpha = 0$.

Consider again the Laurent series of f at 0: $f(x) = \frac{a_{-1}}{x} + a_0 + a_1x + x^2\phi(x)$ with $\phi \in \mathcal{M}(\mathbb{K})$ and $\phi(0) \neq \infty$. Then

$$\frac{f'}{f^2} = \frac{\frac{-a_{-1}}{x^2} + a_1 + x\psi(x)}{\frac{(a_{-1})^2}{x^2} + \frac{2a_0a_1}{x} + a_0^2 + 2a_1a_{-1} + x\xi(x)}$$

where both $\psi,\ \xi \in \mathcal{M}(\mathbb{K})$ have no pole at 0. Clearly, $\frac{f'}{f^2}$ is analytic at 0 and its value is $\frac{-1}{a_{-1}}$. But since 0 is a zero of $\frac{f'}{f^2} + b$, we have $a_{-1} = \frac{1}{b}$, what is excluded by hypothesis. Thus we have proved that every zero of $\frac{f'}{f^2} + b$ is a zero of $f' + bf^2$ (that is not a zero of f) with the same multiplicity and this ends the proof of Theorem 9.

Theorem 10: *Let $b \in \mathbb{K}^*$ and let $f \in \mathcal{M}(\mathbb{K})$ have finitely many zeros and finitely many residues at its simple poles equal to $\frac{1}{b}$ and be such that $f' + bf^2$ has finitely many zeros. Then f belongs to $\mathbb{K}(x)$.*

Proof: Let $f = \frac{P}{l}$ with $P \in \mathbb{K}[x]$, $l \in \mathcal{A}(\mathbb{K})$ having no common zero with P. Then

$f' + bf^2 = \dfrac{P'l - l'P + bP^2}{l^2}$. By hypothesis, this function has finitely many zeros. Moreover, if a is a zero of $P'l - l'P + bP^2$ but is not a zero of $f' + bf^2$, then by Lemma 7 it is a pole of order 1 of f such that $\text{res}(f, a) = \dfrac{1}{b}$. Consequently, $P'l - l'P + bP^2$ has finitely many zeros and hence, we may write $\dfrac{P'l - l'P + bP^2}{l^2} = \dfrac{Q}{l^2}$ with $Q \in \mathbb{K}[x]$, hence $P'l - l'P = -bP^2 + Q$. But then, by Theorem D, l is a polynomial, which ends the proof.

Remark: If $f(x) = \dfrac{1}{x}$, the function $f' + bf^2$ has no zero whenever $b \neq 1$.

Theorem 11: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many zeros of order ≥ 2 and let $b \in \mathbb{K}$. Then $\dfrac{f'}{f^2} + b$ has infinitely many zeros. Moreover, if $b \neq 0$, every zero α of $\dfrac{f'}{f^2} + b$ that is not a zero of $f' + bf^2$ is a pole of f of order 1 such that the residue of f at α is equal to $\dfrac{1}{b}$.*

Proof: Let $g = \dfrac{f'}{f^2} + b$. Since all zeros of f are of order 1 except maybe finitely many, g has finitely many poles of order ≥ 3, hence a primitive G of g has finitely many poles of order ≥ 2. Consequently, by Theorem E, g has infinitely many zeros.

Now, suppose $b \neq 0$. Let α be a zero of g. If α is not a pole of f, it is a zero of $f' + bf^2$ and we can see that it is not a zero of f.

Finally, suppose that α is a pole of f. Then it must be a pole of order 1 and then, by Lemma 7, the residue of f at α is $\frac{1}{b}$.

Corollary 11.1: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have finitely many zeros of order ≥ 2 and finitely many poles of order 1 and let $b \in \mathbb{K}^*$. Then $f' + bf^2$ has infinitely many zeros that are not zeros of f.*

Remarks: As noticed above, in Archimedean analysis, the typical example of a meromorphic function f such that $f' - f^2$ has no zero is $\tan(x)$ and its residue is -1 at each pole of f. Here we find the same implication but we can't find an example satisfying such properties.

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