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► To cite this version:

Alain Escassut, José-Luis Riquelme. Applications of branched values to p-adic functional equations on analytic functions. p-Adic Numbers, Ultrametric Analysis and Applications, 2014. hal-01920281

HAL Id: hal-01920281

<https://hal.uca.fr/hal-01920281>

Submitted on 13 Nov 2018

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Applications of branched values to p-adic functional equations on analytic functions

Alain Escassut and José-Luis Riquelme

2014/ 03/ 29

Abstract

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. Results on branched values obtained in a previous paper are used to prove that algebraic functional equations of the form $g^q = hf^q + w$ have no solution among transcendental entire functions f , g or among unbounded analytic functions inside an open disk, when w is a polynomial or a bounded analytic function and h is a polynomial or an analytic function whose zeros are of order multiple of q .

We also show that an analytic function whose zeros are multiple of an integer q inside a disk is the q -th power of another analytic function, provided q is prime to the residue characteristic.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, of residue characteristic p , complete with respect to an ultrametric absolute value $|\cdot|$. Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the closed disk $\{x \in \mathbb{K} : |x - \alpha| \leq R\}$ and by $d(\alpha, R^-)$ the open disk $\{x \in \mathbb{K} : |x - \alpha| < R\}$ contained in $\mathbb{K}$, by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$ -algebra of analytic functions in $\mathbb{K}$ (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ and by $\mathbb{K}(x)$ the field of rational functions.

In the same way, given $\alpha \in \mathbb{K}$ and $R > 0$, we denote by $\mathcal{A}(d(\alpha, R^-))$ the $\mathbb{K}$ -algebra of analytic functions in $d(\alpha, R^-)$ (i.e. the set of power series with a radius of convergence $\geq R$) and by $\mathcal{M}(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}(d(\alpha, R^-))$. We then denote by $\mathcal{A}_b(d(\alpha, R^-))$ the $\mathbb{K}$ -algebra of bounded analytic functions in $d(\alpha, R^-)$ and by $\mathcal{M}_b(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, R^-))$. And we set $\mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-))$ and $\mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-))$. As in complex functions, a meromorphic function is said to be *transcendental* if it is not a rational function. Then transcendental functions are known to be transcendental on the field $\mathbb{K}(x)$ [4].

In complex functions theory, a notion closely linked to Picard's exceptional values [4], [6] was introduced: the notion of "perfectly branched value" [2]. In [5] the same notion was introduced on $\mathcal{M}(\mathbb{K})$ and on $\mathcal{M}_u(d(\alpha, R^-))$.

Let us recall these notions.

Definition: Let f be a meromorphic function in $\mathbb{C}$ (resp. $\mathbb{K}$, resp. $d(\alpha, R^-)$). A value $b \in \mathbb{C}$ will be called a *perfectly branched value* for f if all zeros of $f - b$ are of multiple order except finitely many. And b is called a *totally branched value* for f if all zeros of $f - b$ are of multiple order, without exception.

⁰2000 Mathematics Subject Classification: 12J25; 30D35; 30G06; 46S10

⁰Keywords: P-adic meromorphic functions, Nevanlinna's Theory, Values distribution, Branched values, Exceptional values.

Here we want to apply these properties in order to examine certain algebraic functional equations.

In $\mathbb{C}$ it is known that a transcendental meromorphic function admits at most 4 perfectly branched values and an entire function admits at most 2 perfectly branched values [2]. As explained by K. S. Charak in [2], these numbers, respectively 4 and 2, are sharp. Weierstrass function $\wp$ has 4 totally branched values (considering ∞ as a value) and of course, sine and cosine functions admit two totally branched values: 1 and -1 .

On the field $\mathbb{K}$, in [5] it is proven that a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}_u(d(a, R^-))$ has at most 4 perfectly branched values and more precisely, a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ has at most 3 totally branched values. An unbounded analytic function $f \in \mathcal{A}(d(a, R^-))$ has at most 2 perfectly branched values. But it is also proven that a transcendental meromorphic function having finitely many poles $f \in \mathcal{M}(\mathbb{K})$ has at most 1 perfectly branched value.

Notation: Given $f \in \mathcal{A}(d(0, r))$, we put $|f|(r) = \lim_{\substack{|x| \rightarrow r, \\ |x| \neq r}} |f(x)|$.

First, we must recall some classical results:

Theorem A [3], [4]: *Let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}(d(0, R^-))$). Then $|f + g|(r) \leq \max(|f|(r), |g|(r))$, $|f \cdot g|(r) = |f|(r)|g|(r)$, If $|f|(r) > |g|(r)$, then $|f + g|(r) = |f|(r)$.*

Theorem B [3], [4]: *Given $h \in \mathcal{A}(d(0, R^-))$, then*

- i) *h has no zero inside the disk $d(0, R^-)$ if and only if $|h(x) - h(a)| < |h(a)| \forall x \in d(a, R^-)$,*
- ii) *$|h|(r) \leq O(r^t)$ for some $t \in \mathbb{N}$ if and only if $h \in \mathbb{K}[x]$.*

Theorem C [4]: *Let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}(d(a, R^-))$) be different from 0. If $f \cdot g$ belongs to $\mathbb{K}[x]$ (resp. if $f \cdot g$ belong $\mathcal{A}_b(d(a, R^-))$), then both f, g belong to $\mathbb{K}[x]$ (resp. both f, g belong to $\mathcal{A}_b(d(a, R^-))$).*

Theorem D [3], [4]: *Let $q \in \mathbb{N}^*$ be prime to p (resp. let $p = q = 2$). Given an analytic function $w \in \mathcal{A}(d(a, R^-))$ such that $|w(x) - 1| < 1 \forall x \in d(a, R^-)$ (resp. $|w(x) - 1| < \frac{1}{2} \forall x \in d(a, R^-)$), the function $\sqrt[q]{w(x)}$ is defined in $d(a, R^-)$ and belongs to $\mathcal{A}_b(d(a, R^-))$.*

Theorem E [4], [8] : *Suppose $\mathbb{K}$ is spherically complete. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of $d(a, R^-)$ such that $\lim_{n \rightarrow +\infty} |a_n| = R$ and $|a_n| < |a_{n+1}| \forall n \in \mathbb{N}$ and let $(q_n)_{n \in \mathbb{N}}$ be a sequence of positive integers. There exists $f \in \mathcal{A}(d(a, R^-))$ admitting each a_n as a zero of f of order q_n .*

Theorem F [5] : *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many poles or zeros. Then f has at most one perfectly branched value.*

Remark : In [5], Theorem F was stated without mentioning the hypothesis when f has finitely many zeros.

Theorem G is well known and easily proven [4]

Theorem H : *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental. There exists at most one value $b \in \mathbb{K}$ such that $f - b$ has finitely many zeros.*

Theorem I : *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many poles or zeros. Then f has at most one perfectly branched value.*

We want now to generalize the notion of branched value.

Definition: Let $f \in \mathcal{M}(\mathbb{K})$ and let $P \in \mathbb{K}(x)$. The rational function P will be called *a branched rational function for f* if all zeros of $f - P$ are multiple except at most finitely many.

Theorem 1: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many poles or zeros. Then f has at most one perfectly branched rational function.*

Proof. Suppose that f has two distinct perfectly branched rational functions P and Q . Then $f - Q$ has two distinct perfectly branched rational functions: $P - Q$ and 0. So, without loss of generality, we can assume $Q = 0$. Now, let $g = \frac{f}{P}$. We can see that g satisfies all hypotheses of Theorem F and has two perfectly branched values: 0 and 1. Consequently, g is not transcendental and therefore neither is f , a contradiction.

The following Theorem 2 generalizes a Theorem in [1].

Theorem 2: *Let $P, Q, S \in \mathbb{K}(x)$ be non-dentically zero and suppose that two functions $f, g \in \mathcal{M}(\mathbb{K})$ have finitely many poles or zeros and satisfy $P(x)(g(x))^n = Q(x)(f(x))^m + S(x) \forall x \in \mathbb{K}$ with $\min(m, n) \geq 2$. Then f and g belong to $\mathbb{K}(x)$.*

Proof: Let $F(x) = Q(x)(f(x))^m + S(x)$. Then F belongs to $\mathcal{M}(\mathbb{K})$ and has finitely many poles. Since $P(x)(g(x))^n = F(x)$, 0 is a perfectly branched value for F . On the other hand, $F(x) - S(x) = Q(x)(f(x))^m$, hence $S(x)$ is a perfectly branched polynomial for F . But then by Theorem 1, F is not transcendental and hence belongs to $\mathbb{K}(x)$ and then, so does $(f(x))^m$. But since the entire function f is algebraic over $\mathbb{K}(x)$, actually it belongs to $\mathbb{K}(x)$. But then, so does g^n and similarly, so does g .

Corollary 2.1: *Let $P, Q, S \in \mathbb{K}[x]$ be non-dentically zero and suppose that two entire functions $f, g \in \mathcal{A}(\mathbb{K})$ satisfy $P(x)(g(x))^n = Q(x)(f(x))^m + S(x) \forall x \in \mathbb{K}$ with $\min(m, n) \geq 2$. Then f and g belong to $\mathbb{K}[x]$.*

In Theorem 3 we will use the following lemma whose proof is classical [4]:

Lemma 1: *Let $f \in \mathcal{A}(\mathbb{K})$ have all zeros of order multiple of an integer q . Then there exists $\phi \in \mathcal{A}(\mathbb{K})$ such that $f = \phi^q$.*

Proof: Let $(\zeta_n)_{n \in \mathbb{N}}$ be the sequence of zeros, each of order qs_n . By classical results [3], [4], there exists a function $\psi \in \mathcal{A}(\mathbb{K})$ admitting each ζ_n as a zero of order s_n . Consequently, the function $\frac{f}{(\psi)^q}$ is a meromorphic function in $\mathbb{K}$ having no zero and no pole and therefore is a constant λ . Now taking $\alpha \in \mathbb{K}$ such that $\alpha^q = \lambda$ and putting $\phi = \alpha\psi$, we have $f = (\phi)^q$.

Lemma 2: *Let $f, g \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. let $f, g \in \mathcal{A}_u(d(0, R^-))$) with $f \neq g$. Then for every integer $q \geq 2$, if $f^q - g^q$ is not identically zero, it belongs to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. to $\mathcal{A}_u(d(0, R^-))$).*

Proof: Let ζ be a primitive q -th root of 1. We have $f^q - g^q = \prod_{j=0}^{q-1} (f - \zeta^j g)$. By Theorem C, through an immediate induction, we can see that if $f^q - g^q$ belongs to $\mathbb{K}[x]$ (resp. to $\mathcal{A}_b(d(0, R^-))$)

then all factors $(f - \zeta^j g)$ belong to $\mathbb{K}[x]$ (resp. to $\mathcal{A}_b(d(0, R^-))$). But then, $(f - g) + (f - \zeta g)$ belongs to $\mathbb{K}[x]$ (resp. to $\mathcal{A}_b(d(0, R^-))$) and so do f and g .

Theorem 3: *Let $h \in \mathcal{A}(\mathbb{K})$ have all its zeros of order multiple of $q \geq 2$ and let $w \in \mathbb{K}[x]$ be non identically 0. Then the functional equation*

$$(\mathcal{E}) \quad (g(x))^q = h(x)(f(x))^q + w(x)$$

has no solution $f, g \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[X]$. Moreover, if $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$, then $(\mathcal{E})$ has no solution except maybe if f is identically zero.

Proof: Let $f, g \in \mathcal{A}(\mathbb{K})$ satisfy $(\mathcal{E})$. If $h \in \mathbb{K}[x]$, by Corollary 2.1 both f and g are polynomials. Now suppose that $h \notin \mathbb{K}[x]$. By Lemma 1 there exists $\phi \in \mathcal{A}(\mathbb{K})$ such that $\phi^q = h$. Consequently, we have $w(x) = (g(x))^q - (\phi(x)f(x))^q$ and hence, by Lemma 2, both g and $\phi.f$ belong to $\mathbb{K}[x]$. Now, since h is transcendental, so is ϕ . And hence so is $\phi.f$, a contradiction if f is not identically zero.

We can now prove Theorems 4 and 5:

Theorem 4: *Let q be an integer prime to p . Let $R \in]0, +\infty[$ and let $h, w \in \mathcal{A}_b(d(0, R^-))$ be such that $|h(x) - h(0)| < |h(0)| \forall x \in d(0, R^-)$. Then the equation $(\mathcal{E})$ $g^q = hf^q + w$ has no solution $f, g \in \mathcal{A}_u(d(0, R^-))$.*

Theorem 5: *Suppose $\mathbb{K}$ has residue characteristic 2. Let $R \in]0, +\infty[$ and let $h, w \in \mathcal{A}_b(d(0, R^-))$ be such that $|h(x) - h(0)| < \frac{|h(0)|}{2} \forall x \in d(0, R^-)$. Then the equation $(\mathcal{E})$ $g^2 = hf^2 + w$ has no solution $f, g \in \mathcal{A}_u(d(0, R^-))$.*

Proof of Theorems 4 and 5: Suppose that $(\mathcal{E})$ has solutions $f, g \in \mathcal{A}_u(d(0, R^-))$. By Theorem D, in both theorems we can apply the root function $\sqrt[q]{\cdot}$ to the function h and therefore there exists $\sigma \in \mathcal{A}_b(d(0, R^-))$ such that $\sigma^q = h$. Consequently, we have $g^q - (\sigma f)^q = w$. But by Lemma 2, $g^q - (\sigma f)^q$ is unbounded, a contradiction since w is bounded. This ends the proof.

Corollary 4.1: *Let q be an integer prime to p . Let $R \in]0, +\infty[$ and let $h, w \in \mathcal{A}_b(d(0, R^-))$ be such that $h(x) \neq 0 \forall x \in d(0, R^-)$. Then the equation $(\mathcal{E})$ $g^q = hf^q + w$ has no solution $f, g \in \mathcal{A}_u(d(0, R^-))$.*

Proof: Indeed, since h has no zero in $d(a, R^-)$, by Theorem B it satisfies $|h(x) - h(a)| < |h(a)| \forall x \in d(0, R^-)$.

In the proof of Theorem 6 we will need the following Lemma 3;

Lemma 3: *Let $q \in \mathbb{N}^*$ and let L be a complete algebraically closed extension of $\mathbb{K}$. Let $f \in \mathcal{A}(d(a, R^-))$ and suppose that there exists a power series g with coefficients in L , with radius of convergence R such that $(g(x))^q = f(x) \forall x \in d(a, R^-)$. Then g has all coefficients in $\mathbb{K}$ and belongs to $\mathcal{A}(d(a, R^-))$.*

Proof: Without loss of generality we can obviously suppose $a = 0$. Let $f(x) = \sum_{n=0}^{+\infty} b_n x^n$ (with

$b_n \in \mathbb{K}$) and let $g(x) = \sum_{n=0}^{+\infty} a_n x^n$. Then $(a_0)^q = b_0$, hence $a_0 \in \mathbb{K}$ because $\mathbb{K}$ is algebraically closed.

Now suppose we have proven that $a_n \in \mathbb{K} \forall n \leq t-1$. We can see that b_t is of the form $a_t(a_0)^{q-t} + z$ where z is a polynomial in $a_0, a_1, \dots, a_{t-1}$. Therefore a_t also belongs to $\mathbb{K}$. Consequently, g has all coefficients in $\mathbb{K}$, which ends the proof.

Theorem 6 applies to functions inside a disk. It is similar to Lemma 1 for functions in $\mathbb{K}$ but is more delicate.

Theorem 6: *Let $q \in \mathbb{N}^*$ be prime to p . Let $f \in \mathcal{A}(d(a, R^-))$ and suppose that all zeros of f are of order multiple of q . Then there exists a function $g \in \mathcal{A}(d(a, R^-))$ such that $f(x) = (g(x))^q \forall x \in d(a, R^-)$.*

Proof: Suppose first that $\mathbb{K}$ is spherically complete. Let $(\alpha_n)_{n \in \mathbb{N}}$ be the sequence of zeros of the function h in $d(a, R^-)$, with $\lim_{n \rightarrow +\infty} |\alpha_n| = R$, each being of respective order qs_n .

By Theorem E, there exists $\phi \in \mathcal{A}(d(a, R^-))$ admitting for zeros $(\alpha_n)_{n \in \mathbb{N}}$, each being of respective order s_n . Thus, $\frac{f(x)}{(\phi(x))^q}$ has no zero and no pole and therefore is an invertible bounded

function ℓ which belongs to $\mathcal{A}_b(d(a, R^-))$. Let $\psi(x) = \frac{\ell(x)}{\ell(a)}$ and let λ be a q -th root of $\ell(a)$. Then, since ℓ has no zero in $d(a, R^-)$, we have $|\ell(x) - \ell(a)| < |\ell(a)| \forall x \in d(a, R^-)$ and therefore $|\psi(x) - \psi(a)| = |\psi(x) - 1| < 1$.

Consequently, since p does not divide q , by Theorem D, the function $\sqrt[q]{\cdot}$ applies to $\psi(x)$ in $d(a, R^-)$ and then $\sqrt[q]{\psi(x)}$ belongs to $\mathcal{A}_b(d(a, R^-))$. So, we have $f(x) = (\lambda\phi(x)\sqrt[q]{\psi(x)})^q$, which proves the claim when $\mathbb{K}$ is spherically complete.

Consider now the general case, when $\mathbb{K}$ is no longer supposed to be spherically complete. Let $\widehat{\mathbb{K}}$ be a spherically complete algebraically closed extension of $\mathbb{K}$. Given a disk $d(a, r^-)$ of $\mathbb{K}$, we will denote by $\widehat{d}(a, r^-)$ the similar disk of $\widehat{\mathbb{K}}$: $\{x \in \widehat{\mathbb{K}} \mid |x - a| < r\}$.

The function f has continuation to a function $\widehat{f}$ which belongs to $\mathcal{A}(\widehat{d}(a, R^-))$ and hence there exists a function $g \in \mathcal{A}(\widehat{d}(a, R^-))$ such that $g^q = \widehat{f}$. Then by Lemma 3, g is a power series that has all coefficients in $\mathbb{K}$ and hence belongs to $\mathcal{A}(d(a, R^-))$.

The following Theorems 7 and 8 complete previous results obtained with help of the p -adic Nevanlinna Theory [1], [4].

Theorem 7: *Let $q \in \mathbb{N}^*$ be prime to p . Let $h \in \mathcal{A}(d(a, R^-))$ be such that all zeros of h are of order multiple of q and let $w \in \mathcal{A}_b(d(a, R^-))$. Then the functional equation*

$$(\mathcal{E}) \quad (g(x))^q = h(x)(f(x))^q + w(x)$$

has no solution in $\mathcal{A}_u(d(a, R^-))$.

Proof: Suppose that $(\mathcal{E})$ has solutions $f, g \in \mathcal{A}_u(d(a, R^-))$. By Theorem 4 there exists a function $\phi \in \mathcal{A}(d(a, R^-))$ such that $h(x) = (\phi(x))^q \forall x \in d(a, R^-)$. Consequently we can write $(g(x))^q - (\phi(x)f(x))^q = w(x)$. But by Lemma 2, $(g(x))^q - (\phi(x)f(x))^q$ is unbounded, a contradiction to the hypothesis on w .

In order to prove Theorems 8 we must briefly recall the p -adic Nevanlinna Theory and its Main Theorem on 3 small functions with its corollary for analytic functions [4], [7].

We have to introduce the counting function of zeros and poles of f , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way:

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n .

$$\text{We set } Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|).$$

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$$\overline{Z}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|).$$

In the same way, considering the finite sequence (b_n) , $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n , we put

$$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|).$$

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega}}_0(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega}}_0(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$$\overline{N}(r, f) = \overline{\overline{\omega}}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|).$$

Now we can define the the Nevanlinna function $T(r, f)$ in I or J as

$T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *characteristic function of f* .

Concerning meromorphic functions on $\mathbb{C}$, the definition of $T(r, f)$ is

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| dt + N(r, f)$$

Given f and $w \in \mathcal{M}(\mathbb{K})$ (resp. f and $w \in \mathcal{M}(d(0, R^-))$), resp. f and $w \in \mathcal{M}(\mathbb{C})$, w is called *a small function respectively to f* if $\lim_{r \rightarrow +\infty} \frac{T(r, w)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R^-} \frac{T(r, w)}{T(r, f)} = 0$, resp. $\lim_{r \rightarrow +\infty} \frac{T(r, w)}{T(r, f)} = 0$).

Given $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$, resp. $f \in \mathcal{M}(\mathbb{C})$), we denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$, resp. $\mathcal{M}_f(\mathbb{C})$), the set of functions $w \in \mathcal{M}(\mathbb{K})$ (resp. the set of functions $w \in \mathcal{M}(d(0, R^-))$, resp. $w \in \mathcal{M}(\mathbb{C})$) which are small functions respectively to f . Similarly, we denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(0, R^-))$, resp. $\mathcal{A}_f(\mathbb{C})$), the set of functions $w \in \mathcal{A}(\mathbb{K})$ (resp. the set of functions $w \in \mathcal{A}(d(0, R^-))$, resp. $w \in \mathcal{A}(\mathbb{C})$) which are small functions respectively to f .

Remarks: 1) Given $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$), we have $T(r, f) = Z(r, f)$.

2) In $\mathcal{M}(d(0, R^-))$, concerning small functions, obviously given any $f \in \mathcal{M}_u(d(0, R^-))$, all functions $u \in \mathcal{M}_b(d(0, R^-))$ belong to $\mathcal{M}_f(d(0, R^-))$.

In the proof of Theorem 8, we will use the following Lemma L and Theorem N, known as Nevanlinna second main Theorem on 3 small functions.

Lemma J [4]: Let $f, g \in f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$). Then $Z(r, f.g) = Z(r, f) + Z(r, g)$ and $Z(r, f + g) \leq \max(Z(r, f), Z(r, g)) + O(1)$. Let $f, g \in f \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(0, R^-))$). Then $T(r, f.g) \leq T(r, f) + T(r, g)$ and $T(r, f + g) \leq T(r, f) + T(r, g)$.

The following Lemma L is an obvious consequence:

Lemma L [4]: *Given $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$), then $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$).*

Lemma M [4]: *Let $f \in \mathcal{A}(d(0, R^-))$. Then f belongs to $\mathcal{A}_b(d(0, R^-))$ if and only if $T(r, f)$ is bounded when r tends to R .*

Theorem N.1 [6]: *Let $f \in \mathcal{A}(\mathbb{C})$ and let $w_1, w_2, w_3 \in \mathcal{A}_f(\mathbb{C})$. Then $T(r, f) \leq \sum_{j=1}^3 \overline{Z}(r, f - w_j) + o(T(r, f))$.*

Theorem N.2 [4], [7]: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$) and let $w \in \mathcal{A}_f(\mathbb{K})$ (resp. $w \in \mathcal{A}_f(d(0, R^-))$). Then $T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - w) + o(T(r, f))$.*

We can now state and prove Theorem 8

Theorem 8: *Let $h, w \in \mathcal{A}_b(d(a, R^-))$ and let $m, n \in \mathbb{N}^*$ be such that $\min(m, n) \geq 2$, $\max(m, n) \geq 3$. Then the functional equation*

$$(\mathcal{E}) \quad (g(x))^n = h(x)(f(x))^m + w(x)$$

has no solution in $\mathcal{A}_u(d(a, R^-))$.

Proof: Let $F(x) = g(x)^n$. Thanks to Corollary N2, we can write

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - w) + o(T(r, F)).$$

Now, it appears that $\overline{Z}(r, F) \leq \frac{1}{n} Z(r, F)$. Moreover, since h is bounded, by Lemma M $Z(r, h)$ is bounded, hence by Lemma L, we have $\overline{Z}(r, hf^m) \leq Z(r, f) + Z(r, h) = Z(r, f) + O(1)$, therefore

$$\overline{Z}(r, hf^m) \leq \frac{1}{m} Z(r, hf^m) + O(1) = \frac{1}{m} Z(r, F) + O(1).$$

On the other hand, $Z(r, F) = Z(r, F - w) + O(1) = T(r, F) + O(1)$. Consequently, by (1), we can derive

$$T(r, F) \leq \left(\frac{1}{m} + \frac{1}{n}\right) T(r, F) + o(T(r, F)).$$

Therefore we have $\frac{1}{m} + \frac{1}{n} \geq 1$, a contradiction to the hypothesis which implies $\frac{1}{m} + \frac{1}{n} \leq \frac{5}{6}$.

We will now examine a kind of generalization of Theorem H: By Theorem N1, Theorem 9 is immediate:

Theorem 9: *Let $f \in \mathcal{M}(\mathbb{C}) \setminus \mathbb{C}(x)$. There exist at most two functions $w_1, w_2 \in \mathcal{M}_f(\mathbb{C})$ such that $f - w_j$ have finitely many zeros ($j = 1, 2$). Moreover, if $f \in \mathcal{A}(\mathbb{C}) \setminus \mathbb{C}[x]$ then there exists at most one function $w \in \mathcal{A}_f(\mathbb{C})$ such that $f - w$ has finitely many zeros.*

Proof: Suppose that there exist three functions $w_j \in \mathcal{M}_f(\mathbb{C})$ such that $f - w_j$ has finitely many zeros $1 \leq j \leq 3$. Then $\sum_{j=1}^3 \overline{Z}(r, f - w_j)$ admits an upper bound of the form $q \log(r) + O(1)$. Consequently, by Theorem N1, we have $T(r, f) \leq q \log(r) + O(1)$, a contradiction since f is transcendental. If $f \in \mathcal{A}(\mathbb{C}) \setminus \mathbb{C}[x]$, we can directly apply Theorem N with $N(r, f) = 0$.

Theorem 10: Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $f \in \mathcal{M}_u(d(0, R^-))$). There exists at most one function $w \in \mathcal{M}_f(\mathbb{K})$, (resp. $w \in \mathcal{M}_f(d(0, R^-))$) such that $f - w$ has finitely many zeros.

Proof: Suppose that there exist two distinct functions $g_1, g_2 \in \mathcal{M}_f(\mathbb{K})$, (resp. $g_1, g_2 \in \mathcal{M}_f(d(0, R^-))$) such that $f - g_k$ has finitely many zeros. So, there exist $P_1, P_2 \in \mathbb{K}[x]$ and $h_1, h_2 \in \mathcal{A}(\mathbb{K})$ (resp. $h_1, h_2 \in \mathcal{A}(d(0, R^-))$) such that $f - g_k = \frac{P_k}{h_k}$, $k = 1, 2$ and hence we notice that

$$(1) \quad T(r, f) = T(r, \frac{P_k}{h_k}) + o(T(r, f)) = T(r, h_k) + o(T(r, f)) \quad k = 1, 2.$$

Consequently, putting $g = g_2 - g_1$, we have $\frac{P_1}{h_1} = \frac{P_2}{h_2} + g$ and of course, by Lemma L, g belongs to $\mathcal{M}_f(\mathbb{K})$ (resp. to $\mathcal{M}_f(d(0, R^-))$). Therefore $P_1 h_2 - P_2 h_1 = g h_1 h_2$ and hence

$$(2) \quad T(r, P_1 h_2 - P_2 h_1) = T(r, g h_1 h_2).$$

Now, by Lemma J we have

$$T(r, P_1 h_2 - P_2 h_1) \leq \max(T(r, P_1 h_2), T(r, P_2 h_1)) \leq \max(T(r, h_1), T(r, h_2)) + o(T(r, f))$$

and hence by (1), we obtain

$$(3) \quad T(r, P_1 h_2 - P_2 h_1) \leq T(r, f) + o(T(r, f)).$$

On the other hand, by Lemma J, we have $T(r, g h_1 h_2) \geq T(r, h_1 h_2) - T(r, g)$. But by Lemma J again, we have $T(r, h_1 h_2) = T(r, h_1) + T(r, h_2) = 2T(r, f) + o(T(r, f))$. Consequently, (2) and (3) yield $2T(r, f) + o(T(r, f)) \leq T(r, f) + o(T(r, f))$, a contradiction.

Remark: Concerning complex meromorphic functions, it is well known that a meromorphic function f in the whole field $\mathbb{C}$ can have at most two exceptional or quasi-exceptional values (i.e. values b such that $f - b$ has finitely many zeros). Thanks to the complex Nevanlinna Theorem on 3 small functions [6], we can easily show that there exist at most two small functions g (with respect to f) such that $f - g$ has finitely many zeros.

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