Complex and p-adic meromorphic functions $f', P'(f), g', P'(g)$ sharing a small function

Alain Escassut, Kamal Boussaf, Jacqueline Ojeda

► To cite this version:

Alain Escassut, Kamal Boussaf, Jacqueline Ojeda. Complex and p-adic meromorphic functions $f', P'(f), g', P'(g)$ sharing a small function. International Journal of Pattern Recognition and Image Analysis, Advances in Mathematical Theory and Applications (PRIA), 2014. hal-01920265

HAL Id: hal-01920265

<https://hal.uca.fr/hal-01920265>

Submitted on 13 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Complex and p -adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function

Alain Escassut, Kamal Boussaf and Jacqueline Ojeda¹

November 3, 2013

Abstract

Let $\mathbb{K}$ be a complete algebraically closed p -adic field of characteristic zero. We apply results in algebraic geometry and a new Nevanlinna theorem for p -adic meromorphic functions in order to prove results of uniqueness in value sharing problems, both on $\mathbb{K}$ and on $\mathbb{C}$. Let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or $\mathbb{C}$ or in an open disk. Let f, g be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or in $\mathbb{C}$ or meromorphic functions in an open disk of $\mathbb{K}$ that are not quotients of bounded analytic functions. We show that if $f'P'(f)$ and $g'P'(g)$ share a small function α counting multiplicity, then $f = g$, provided that the multiplicity order of zeros of P' satisfy certain inequalities. A breakthrough in this paper consists of replacing inequalities $n \geq k + 2$ or $n \geq k + 3$ used in previous papers by Hypothesis (G). In the p -adic context, another consists of giving a lower bound for a sum of q counting functions of zeros with $(q - 1)$ times the characteristic function of the considered meromorphic function.

Notation and definition: Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value $|\cdot|$. We will denote by $\mathbb{E}$ a field that is either $\mathbb{K}$ or $\mathbb{C}$. Throughout the paper we denote by a a point in $\mathbb{K}$. Given $R \in]0, +\infty[$ we define disks $d(a, R) = \{x \in \mathbb{K} \mid |x - a| \leq R\}$ and disks $d(a, R^-) = \{x \in \mathbb{K} \mid |x - a| < R\}$.

A polynomial $Q(X) \in \mathbb{E}[X]$ is called a *polynomial of uniqueness for a family of functions $\mathcal{F}$ defined in a subset of $\mathbb{E}$* if $Q(f) = Q(g)$ implies $f = g$. The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [11], [13], [20] for complex functions and in [1], [2], [9], [10], [17], [18], for p -adic functions.

Throughout the paper we will denote by $P(X)$ a polynomial in $\mathbb{E}[X]$ such that $P'(X)$ is of the form $b \prod_{i=1}^l (X - a_i)^{k_i}$ with $l \geq 2$ and $k_1 \geq 2$. The polynomial P will be said to *satisfy Hypothesis (G)* if $P(a_i) + P(a_j) \neq 0 \forall i \neq j$.

We will improve the main theorems obtained in [5] and [6] with the help of a new hypothesis denoted by Hypothesis (G) and by thoroughly examining the situation with p -adic and complex analytic and meromorphic functions in order to avoid a lot of exclusions. Moreover, we will prove a new theorem completing the 2nd Main Theorem for p -adic meromorphic functions. Thanks to this new theorem we will give more precisions in results on value-sharing problems.

⁰2000 Mathematics Subject Classification: 12J25; 30D35; 30G06.

⁰Keywords: Meromorphic, Nevanlinna, Sharing Value, Unicity, Distribution of values.

¹Partially funded by the research project CONICYT ("Inserción de nuevos investigadores en la academia" N°79090014) from the Chilean Government.

Notation: Let L be an algebraically closed field, let $P \in L[x] \setminus L$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c . We denote by $\Phi(P)$ its cardinal.

We denote by $\mathcal{A}(\mathbb{E})$ the $\mathbb{E}$ -algebra of entire functions in $\mathbb{E}$, by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in $\mathbb{E}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{E})$ and by $\mathbb{E}(x)$ the field of rational functions. Throughout the paper, we denote by $\mathcal{A}(d(a, R^-))$ the $\mathbb{K}$ -algebra of analytic functions in $d(a, R^-)$ i.e. the $\mathbb{K}$ -algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a, R^-)$ and we denote

by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$ -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$. We denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$ and finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Theorem O1 [9]: Let $P(X) \in \mathbb{K}[X]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and for $\mathcal{A}_u(d(a, R^-))$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$.

Let $P(X) \in \mathbb{C}[X]$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

Concerning polynomials such that P' has exactly two distinct zeros, we know other results:

Theorem O2 [1], [2], [18]: Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1, c_2\} \geq 2$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.

Theorem O3 [9], [17]: Let $P \in \mathbb{K}[x]$ be of degree $n \geq 6$ be such that P' only has two distinct zeros, one of them being of order 2. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$.

Theorem O4 [18]: Let $P \in \mathbb{C}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1, c_2\} \geq 2$ and $\max(c_1, c_2) \geq 3$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

In order to state theorems and recall the definition of a small function, we must recall the definition of the classical Nevanlinna functions both on a p -adic field and on the field $\mathbb{C}$ together with a few specific properties of ultrametric analytic or meromorphic functions [7], [11], [13].

Notation: Let $\log$ be a real logarithm function of base $b > 1$ and let $\log^+(x) = \max(0, \log(x))$. Let $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(h) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$. These definitions of Nevanlinna's functions are equivalent to those defined in [7].

We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$, counting multiplicities, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the counting function of zeros of f in $d(0, r)$, ignoring multiplicities, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else.

In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp. $\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicities (resp. ignoring multiplicities).

For $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-))$, we call *Nevanlinna function of f* the function $T(r, f) = \max\{Z(r, f), N(r, f)\}$.

Consider now a function $f \in \mathcal{M}(\mathbb{C})$. We can define a function $m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ and we call *Nevanlinna function of f* the function $T(r, f) = m(r, f) + N(r, f)$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{E})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{E})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) is called a *small function with respect to f* , if it satisfies

$$\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0 \quad \left(\text{resp.} \quad \lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0 \right).$$

We denote by $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in $\mathbb{E}$ (resp. in $d(0, R^-)$).

Remark 1. Thanks to classical properties of the Nevanlinna function $T(r, f)$ with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ and $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proven that $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$) and that $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{E})$ (resp. of $\mathcal{M}_f(d(0, R^-))$) [10].

Let us remember the following definition.

Definitions. Let $f, g, \alpha \in \mathcal{M}(\mathbb{E})$ (resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))$). We say that f and g *share the function α C.M.*, if $f - \alpha$ and $g - \alpha$ have the same zeros with the same multiplicities in $\mathbb{E}$ (resp. in $d(0, R^-)$).

In [5] and [6] we have obtained this general Theorem (where results of [5] and [6] here are gathered):

Theorem T: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp for $\mathcal{M}(\mathbb{C})$, resp. for $\mathcal{M}(d(0, R^-))$) with $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} k_1 &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ k_1 &\geq k + 2 \quad (\text{resp. } k_1 \geq k + 3, \text{ resp. } k_1 \geq k + 3) \\ \text{if } l = 2, &\text{ then } k_1 \neq k + 1, \quad 2k, \quad 2k + 1, \quad 3k + 1, \\ \text{if } l = 3, &\text{ then } k_1 \neq k + 1, \quad 2k + 1, \quad 3k_i - k \quad \forall i = 2, 3. \end{aligned}$$

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{E})$ (resp. $f, g \in \mathcal{M}_u((d(a, R^-))$) be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{E}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

In the field $\mathbb{K}$, several particular applications were given when the small function is a constant or a Moebius function. On $\mathbb{C}$, we can't get similar refinements because the complex Nevanlinna Theory is less accurate than the p -adic Nevanlinna Theory.

In the present paper, thanks to the new Hypothesis (G) introduced below, we mean to avoid the hypothesis $k_1 \geq k + 2$ for $\mathcal{M}(\mathbb{K})$ and $k_1 \geq k + 3$ for $\mathcal{M}(\mathbb{C})$ and for $\mathcal{M}(d(a, R^-))$.

But first, we have a new theorem for p -adic analytic functions:

First we can improve results of [5] concerning p -adic analytic functions.

Theorem 1. Let $P(X) \in \mathbb{K}[X]$ be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}_u(d(a, R^-))$) and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. let $f, g \in \mathcal{A}_u(d(a, R^-))$), be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if f, g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Corollary 1.1: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$ and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Example: Let $c \in \mathbb{K}$ be a solution of the algebraic equation:

$$X^{11} \left(\frac{1}{11} - \frac{1}{10} \right) - X^9 \left(\frac{1}{9} - \frac{1}{8} \right) + X \left(\frac{1}{10} - \frac{1}{8} \right) - \frac{1}{11} + \frac{1}{9} = 0.$$

Let

$$P(X) = \frac{X^{11}}{11} - \frac{cX^{10}}{10} - \frac{X^9}{9} + \frac{cX^8}{8}$$

Then we can check that $P'(X) = X^7(X - 1)(X + 1)(X - c)$,

$P(1) = P(c) \neq 0$ and that $P(1) \neq 0$, $P(-1) \neq 0$, $P(1) + P(-1) = c(\frac{1}{4} - \frac{1}{5})$ and $P(-1) - P(1) = 2(\frac{1}{11} - \frac{1}{9})$, hence $P(-1) \neq P(c)$. Consequently, we can apply Corollary 1.1 and show that if $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$, then $f = g$.

Remark 2. Recall Hypothesis (F) due to H. Fujimoto [12]. A polynomial Q is said to satisfy Hypothesis (F) if the restriction of Q to the set of zeros of Q' is injective. In the last example, we may notice that Hypothesis (F) is not satisfied by P .

Corollary 1.2: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 3$ and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}_u(d(a, R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$.

Corollary 1.3: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$ (resp. $\Phi(P) \geq 3$) and let $P'(X) = bX^n \prod_{i=2}^l (X - a_i)$ with $l \geq 3$ and let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}_u(d(a, R^-))$) be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $n \geq l + 3$ then $f = g$. Moreover, if f, g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $n \geq l + 2$ then $f = g$.

In order to improve results of [5] on p -adic meromorphic functions and of [6] on complex meromorphic functions, we have to state Propositions P1 and P2 derived from results of [3] and [4].

Notation and definition: Henceforth we assume that $P(a_1) = 0$ and that $P'(X)$ is of the form $b \prod_{i=1}^l (X - a_i)^{k_i}$ with $l \geq 2$. The polynomial P will be said to satisfy Hypothesis (G) if $P(a_i) + P(a_j) \neq 0 \ \forall i \neq j$.

Proposition P1: Let $P \in \mathbb{K}[X]$ satisfy Hypothesis (G) and $n \geq 2$ (resp. $n \geq 3$). If meromorphic functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{K}^*$, $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(a, R^-)$)) then both f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a, R^-))$).

Proposition P2: Let $P \in \mathbb{C}[X]$ satisfy Hypothesis (G) and $n \geq 3$. If meromorphic functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{C}^*$, $\forall x \in \mathbb{C}$) then both f and g are constant.

From [5] and thanks to Propositions P1 we can now derive the following Theorems 2, 3, 4, 5:

Theorem 2. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp for $\mathcal{M}(d(0, R^-))$) with $l \geq 2$, let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = \max(0, u_5 - 3)$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m),$$

either $k_1 \geq k + 2$ (resp. $k_1 \geq k + 3$, resp. $k_1 \geq k + 3$) or P satisfies Hypothesis (G),
if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}$, $k_1 \neq k + 1$, $2k + 1$, $3k_i - k \ \forall i = 2, 3$.

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}((d(a, R^-))$) be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark 3. the sum $\sum_{m=5}^{\infty} s_m$ is obviously finite.

Corollary 2.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and hypothesis (G), let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

if $l = 3$, then $k_1 \neq \frac{k}{2}$, $k + 1$, $2k + 1$, $3k_i - k \ \forall i = 2, 3$,

if $l \geq 4$, then $n \neq k + 1$. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{20}}{20} - \frac{X^{19}}{19} - \frac{4X^{18}}{18} + \frac{4X^{17}}{17} \\ + \frac{6X^{16}}{16} - \frac{6X^{15}}{15} - \frac{4X^{14}}{14} + \frac{4X^{13}}{13} + \frac{X^{12}}{12} - \frac{X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X - 1)^5(X + 1)^4$ and

$$P(0) = 0, \quad P(1) = \sum_{j=0}^4 C_4^j (-1)^j \left(\frac{1}{10 + 2j} - \frac{1}{9 + 2j} \right),$$

$$P(-1) = - \sum_{j=0}^4 C_4^j \left(\frac{1}{10 + 2j} + \frac{1}{9 + 2j} \right)$$

Consequently, we have $\Phi(P) = 3$ and we check that Hypothesis (G) is satisfied. Now, let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In that example, we have $k_1 = 10$, $k = 9$. Applying our previous work, a conclusion would have required $n \geq k + 2 = 11$.

Theorem 3. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, with $l \geq 2$, let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = u_5 - 3$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m)$$

either $k_1 \geq k + 3$ or P satisfies Hypothesis (G),

if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}, k_1 \neq k + 1, 2k + 1, 3k_i - k \forall i = 2, 3$.

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 3.1 Let $P \in \mathbb{C}[X]$ satisfy $\Phi(P) \geq 4$ and Hypothesis (G), let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m)$$

$k_1 \neq k + 1$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

As noticed in [5], if f, g belong to $\mathcal{M}(\mathbb{K})$ and if α is a constant or a Moebius function, we can get a more accurate statement:

Theorem 4. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$,

$l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

either $k_1 \geq k + 2$ or P satisfies (G)

if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}, k + 1, 2k + 1, 3k_i - k \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem O1, we can derive Corollary 4.1.

Corollary 4.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

either $n \geq k + 2$ or P satisfies (G),

if $l = 3$, then $k_1 \neq \frac{k}{2}, k + 1, 2k + 1, 3k_i - k \ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem O2 we have Corollary 4.2.

Corollary 4.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $k \leq n$, $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k),$$

either $n \geq k + 2$ or P satisfies (G),

$$n \neq k + 1, 2k, 2k + 1, 3k + 1,$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 5. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 4)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 3)$. Suppose P satisfies the following conditions:

either $k_1 \geq k + 2$ or P satisfies (G)

$$k_1 \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

$$k_1 \neq k + 1$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem O1, we can derive Corollary 5.1.

Corollary 5.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$k_1 \geq k + 2$ or P satisfies Hypothesis (G),

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem O2, we have Corollary 5.2

Corollary 5.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$n \geq 9 + \max(0, 5 - k)$,
 either $n \geq k + 2$ or P satisfies (G)
 $n \neq k + 1$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{24}}{24} - \frac{10X^{23}}{23} + \frac{36X^{22}}{22} - \frac{40X^{21}}{21} - \frac{74X^{20}}{20} + \frac{226X^{19}}{19} \\
- \frac{84X^{18}}{18} - \frac{312X^{17}}{17} + \frac{321X^{16}}{16} + \frac{88X^{15}}{15} \\
- \frac{280X^{14}}{14} + \frac{48X^{13}}{13} + \frac{80X^{12}}{12} - \frac{32X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X - 2)^5(X + 1)^4(X - 1)^4$. Next, we have $P(2) < -134378$, $P(1) \in]-2, 11[$, $P(-1) \in]2, 18[$, $19[$. Therefore, $P(0)$, $P(1)$, $P(-1)$, $P(2)$ are all distinct, hence $\Phi(P) = 4$. Moreover, Hypothesis (G) is satisfied.

Now, let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(d(a, R^-))$), resp. let $f, g \in \mathcal{M}(\mathbb{C})$ and let $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $\alpha \in \mathcal{M}(d(a, R^-))$), resp. let $\alpha \in \mathcal{M}(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Particularly, when f, g are entire functions in $\mathbb{C}$ we can simplify the hypothesis:

Theorem 6. Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ with $l \geq 2$ and $k_i \geq k_{i+1}$, $1 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = \max(0, u_5 - 3)$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

either $k_1 \geq k + 2$ or P satisfies hypothesis (G)

$$k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 3, \sum_{m=5}^{\infty} s_m).$$

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Proposition P2, we have Corollaries 6.1 and 6.2:

Corollary 6.1 Let $P \in \mathbb{C}[X]$, let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $k_i \geq k_{i+1}$, $1 \leq i \leq l - 1$ and

let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $t_5 = u_5 - 3$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $t_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions

either $k_1 \geq k + 2$ or P satisfies hypothesis (G)

$$k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 3, \sum_{m=5}^{\infty} s_m)$$

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 6.2 *Let $P \in \mathbb{C}[X]$ and let $P' = b(X - a_1)^n(X - a)^k$ with $\min(k, n) \geq 2$ and $\max(n, k) \geq 3$. Suppose that P satisfies $n \geq 5 + \max(0, 5 - k)$,*

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{11}}{11} + \frac{5X^{10}}{10} + \frac{10X^9}{9} + \frac{10X^8}{8} + \frac{5X^7}{7} + \frac{X^6}{6}.$$

Then $P'(X) = X^5(X + 1)^5$. We can apply Corollary 6.2: given $f, g \in \mathcal{A}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

1 The proofs:

Notation: As usual, given a function $f \in \mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$), we denote by $S_f(r)$ a function of r defined in $]0, +\infty[$ (resp. in $]0, R[$) such that $\lim_{r \rightarrow +\infty} \frac{S_f(r)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{S_f(r)}{T(r, f)} = 0$)

We must recall the classical Nevanlinna Main Theorem:

Theorem N1: [7], [12] *Let $a_1, \dots, a_n \in \mathbb{K}$ (resp. $a_1, \dots, a_n \in \mathbb{K}$, resp. $a_1, \dots, a_n \in \mathbb{C}$) with $n \geq 2, n \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$, resp. let $f \in \mathcal{M}(\mathbb{C})$). Let $S = \{a_1, \dots, a_n\}$. Then, for $r > 0$ we have*

$$(n-1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) - \log r + O(1),$$

resp.

$$(n-1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) + O(1),$$

resp.

$$(n-1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) + S_f(r).$$

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Theorem N2: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. let $f \in \mathcal{A}(d(0, R^-))$, resp. let $f \in \mathcal{A}(\mathbb{C})$) and let $u \in f \in \mathcal{A}_f(\mathbb{K})$ (resp. let $u \in \mathcal{A}_f(d(0, R^-))$, resp. $u \in f \in \mathcal{A}_f(\mathbb{C})$). Then $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - u) + S_f(r)$.*

In order to prove Theorem N3 we need additional lemmas:

Notation: Let $f \in \mathcal{M}(d(a, R^-))$, and let $r \in]0, R[$. By classical results [8], [10] we know that $|f(x)|$ has a limit when $|x|$ tends to r , while being different from r .

We set $|f|(r) = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|$.

Lemma 1. *For every $r \in]0, R[$, the mapping $|\cdot|(r)$ is an ultrametric multiplicative norm on $\mathcal{M}(d(0, R^-))$.*

The following Lemma 2 is the p -adic Schwarz formula:

Lemma 2. *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$) and let $r', r'' \in]0, +\infty[$ (resp. let $r', r'' \in]0, R[$) satisfy $r' < r''$. Then $\log(|f|(r'')) - \log(|f|(r')) = Z(r'', f) - Z(r', f)$.*

Lemma 3. *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Suppose that there exists $a \in \mathbb{K}$ and a sequence of intervals $I_n = [u_n, v_n]$ such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} u_n = R$) and $\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} qT(r, f) - Z(r, f - a) \right) = +\infty$. Set $L = \bigcup_{n=0}^{+\infty} I_n$. Let $b \in \mathbb{K}$, $b \neq a$. Then $Z(r, f - b) = T(r, f) + O(1) \forall r \in L$.*

Proof: We know that the Nevanlinna functions of a meromorphic function f are the same in $\mathbb{K}$ and in an algebraically closed complete extension of $\mathbb{K}$ whose absolute value extends that of $\mathbb{K}$. Consequently, without loss of generality, we can suppose that $\mathbb{K}$ is spherically complete because we know that such a field does admit a spherically complete algebraically closed extension whose absolute value expands that of $\mathbb{K}$. If f belongs to $\mathcal{M}(\mathbb{K})$, we can obviously set it in the form $\frac{g}{h}$ where g, h belong to $\mathcal{A}(\mathbb{K})$ and have no common zero. Next, since $\mathbb{K}$ is supposed to be spherically complete, if f belongs to $\mathcal{M}(d(0, R^-))$ we can also set it in the form $\frac{g}{h}$ where g, h belong to $\mathcal{A}(d(0, R^-))$ and have no common zero [8], [10]. Consequently, we have $T(r, f) = \max(Z(r, g), Z(r, h))$.

By hypothesis we have $\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - a) \right) = +\infty$ i.e.

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - a) \right) = +\infty \text{ i.e.}$$

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} \max(Z(r, g), Z(r, h)) - Z(r, g - ah) \right) = +\infty.$$

Set $B_n = \inf_{r \in I_n} \max(Z(r, g), Z(r, h)) - Z(r, g - ah)$. Since the sequence B_n tends to $+\infty$, clearly, by Lemma 2, the sequence (D_n) defined as $D_n = \sup_{r \in I_n} \left(\frac{|g - ah|(r)}{\max(|g|(r), |h|(r))} \right)$ tends to zero.

Therefore, by Lemma 1 we have $|g|(r) = |ah|(r)$ in I_n when n is big enough. Consequently, by Lemma 2 we have $Z(r, g) = Z(r, ah) + O(1) \forall r \in L$ and hence $T(r, f) = Z(r, h) + O(1) = Z(r, g) + O(1) \forall r \in L$.

Now, consider $g - bh = g - ah + (a - b)h$. By hypothesis we have

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} Z(r, h) - Z(r, g - ah) \right) = +\infty.$$

On the other hand, of course $Z(r, (a - b)h) = Z(r, h) + O(1)$. Consequently, since

$$Z(r, g - bh) = Z(r, g - ah + (a - b)h) \text{ we have } \lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} (Z(r, (a - b)h) - Z(r, g - ah)) \right) = +\infty.$$

Consider now the sequence (E_n) defined as $E_n = \sup_{r \in I_n} \left(\frac{|g - ah|(r)}{|(a - b)h|(r)} \right)$. By Lemma 2, that sequence tends to zero and hence, when r is big enough in L , by Lemma 1 we have $|g - bh|(r) = |a - bh|(r)$. Consequently, when r is big enough in L we have $Z(r, g - bh) = Z(r, bh) = Z(r, h) + O(1)$.

Moreover, we have seen that $Z(r, g) = Z(r, h) + O(1)$ in L , hence $\max(Z(r, g), Z(r, h)) = Z(r, g - bh) + O(1) = \max(Z(r, g - bh), Z(r, h) + O(1))$ i.e. $T(r, f) = T(r, f - b) + O(1)$ in L .

The second Main Theorem is well known in complex and p -adic analysis and is recalled below. But first, we can give here a new theorem of that kind which will be efficient in Theorem 3, 4, 5.

Theorem N3: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then $(q-1)T(r, f) \leq \sum_{j=1}^q Z(r, f - a_j) + O(1)$.*

Proof: Suppose the theorem is wrong. There exists $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and $a_1, \dots, a_q$ such that $(q-1)T(r, f) - \sum_{j=1}^q Z(r, f - a_j)$ admits no superior bound in $]0, +\infty[$. So, there exists a sequence of intervals $J_s = [w_s, y_s]$ such that $w_s < y_s < w_{s+1}$, $\lim_{s \rightarrow +\infty} w_s = +\infty$ (resp. $\lim_{s \rightarrow +\infty} w_s = R$) and

$$\lim_{s \rightarrow +\infty} \left(\inf_{r \in J_s} (q-1)T(r, f) - \sum_{j=1}^q Z(r, f - a_j) \right) = +\infty. \quad (1)$$

Let $M = \bigcup_{s=0}^{\infty} J_s$. For each $j = 1, \dots, q$, we have $Z(r, f - a_j) \leq T(r, f) + O(1)$ in $\mathbb{R}_+$ and hence (1) implies that there exists an index t and a sequence of intervals $I_n = [u_n, v_n]$ included in M , such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} u_n = R$) and

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} (T(r, f) - Z(r, f - a_t)) \right) = +\infty. \quad (2)$$

Let $L = \bigcup_{n=1}^{\infty} I_n$. Then by Lemma 3, in L we have $Z(r, g - a_k h) = T(r, f) + O(1) \forall k \neq t$. Therefore $\sum_{j=1}^q Z(r, f - a_j) \geq (q-1)T(r, f) + O(1)$ in L , a contradiction to (1). Consequently, the Theorem is not wrong.

Remark 4. *Theorem N3 is trivial for analytic functions since by definition, for a function $f \in \mathcal{A}(\mathbb{K})$ or $\mathcal{A}(d(0, R^-))$ we have $T(r, f) = Z(r, f)$. On the other hand, the theorem does not apply to meromorphic functions in $\mathbb{C}$. Indeed, consider a meromorphic function f on $\mathbb{C}$ omitting two values a and b . We have $Z(r, f - a) + Z(r, f - b) = 0$.*

In the proof of Theorems 2- 6 will need the following Lemmas:

Lemma 4. *Let $Q \in \mathbb{K}[x]$ (resp. $Q \in \mathbb{K}[X]$, resp. $Q \in \mathbb{C}[x]$) be of degree n and let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d(0, R^-))$, resp. $f \in \mathcal{M}(\mathbb{C})$) be transcendental. Then $N(r, f') = N(r, f) + \bar{N}(r, f)$, $Z(r, f') \leq Z(r, f) + \bar{N}(r, f) + O(1)$, $nT(r, f) \leq T(r, f'Q(f)) \leq (n+2)T(r, f) - \log r + O(1)$ (resp. $nT(r, f) \leq T(r, f'Q(f)) \leq (n+2)T(r, f) + O(1)$, resp. $nT(r, f) \leq T(r, f'Q(f)) + m(r, \frac{1}{f'}) \leq (n+2)T(r, f) + S_f(r)$).*

Particularly, if $f \in \mathcal{A}(\mathbb{K})$, (resp. $f \in \mathcal{A}(d(0, R^-))$), then $nT(r, f) \leq T(r, f'Q(f)) \leq (n+1)T(r, f) - \log r + O(1)$ (resp. $nT(r, f) \leq T(r, f'Q(f)) \leq (n+1)T(r, f) + O(1)$).

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Lemma 5. *Let $Q(X) \in \mathbb{K}[X]$ and let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(d(0, R^-))$) be such that $Q(f) - Q(g)$ is bounded. Then $f = g$.*

Proof: The polynomial $Q(X) - Q(Y)$ factorizes in the form $(X - Y)F(X, Y)$ with $F(X, Y) \in \mathbb{K}[X, Y]$. Since $Q(f) - Q(g)$ is bounded, so are both factors because the semi-norm $|\cdot|(r)$ is multiplicative on $\mathcal{A}(\mathbb{K})$ (resp. on $\mathcal{A}_u(d(0, R^-))$). Consequently, $f - g$ is a constant c (resp. is a bounded function $u \in \mathcal{A}_b(d(0, R^-))$). Therefore $F(f, g) = F(f, f + c)$ (resp. $F(f, g) = F(f; f + u)$). Let $n = \deg(Q)$. Then we can check that $F(X, X + c)$ is a polynomial in X of degree $n - 1$. Consequently, if $f \in (\mathbb{K})$, $F(f, f + c)$ is a non-constant entire function and therefore is unbounded in $\mathbb{K}$. Similarly, $f \in (d(0, R^-))$, $F(X, X + u)$ is a polynomial in X of degree $n - 1$ with coefficients in $\mathcal{A}(d(0, R^-))$ and therefore $F(f, f + u)$ is unbounded in $d(0, R^-)$, which ends the proof.

Proof of Theorem 1. Without loss of generality, we may assume that $b = 1$. Put $F = f' \prod_{j=1}^l (f - a_j)^{k_j}$ and $G = g' \prod_{j=1}^l (g - a_j)^{k_j}$. Since $f, g \in \mathcal{A}(\mathbb{K})$ and since F and G share α C.M., then $\frac{F - \alpha}{G - \alpha}$ is a meromorphic function having no zeros and no poles in $\mathbb{K}$ (resp. in $d(0, R^-)$), hence it is a constant u in $\mathbb{K} \setminus \{0\}$ (resp. it is an invertible function $u \in \mathcal{A}(d(0, R^-))$).

Suppose $u \neq 1$. Then,

$$F = uG + \alpha(1 - c). \quad (1)$$

Let $r > 0$. Since $\alpha(1 - u) \in \mathcal{A}_f(\mathbb{K})$ (resp. $\alpha(1 - u) \in \mathcal{A}_f(d(0, R^-))$), $\alpha(1 - u)$ obviously belongs to $\mathcal{A}_F(\mathbb{K})$ (resp. to $\mathcal{A}_F(d(0, R^-))$). So, applying Theorem N2 to F , we obtain

$$\begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}(r, F - \alpha(1 - c)) + S_F(r) = \overline{Z}(r, F) + \overline{Z}(G) + S_F(r) \\ &= \sum_{j=1}^l \overline{Z}(r, (f - a_j)^{k_j}) + \overline{Z}(r, f') + \sum_{j=1}^l \overline{Z}(r, (g - a_j)^{k_j}) + \overline{Z}(r, g') + S_f(r) \\ &\leq l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r). \end{aligned}$$

We also notice that if $f, g \in \mathcal{A}(\mathbb{K})$ and if $\alpha \in \mathbb{K}$, we have $T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - \alpha(1 - c)) - \log r + O(1)$ and therefore we obtain

$$T(r, F) \leq l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') - \log r + O(1).$$

Now, let us go back to the general case. Since f is entire, by Lemma 4 we have $T(r, F) = (\sum_{j=1}^l k_j)T(r, f) + Z(r, f') + O(1)$. Consequently, $(\sum_{j=1}^l k_j)T(r, f) \leq l(T(r, f) + T(r, g)) + Z(r, g') + S_f(r)$.

Similarly, $(\sum_{j=1}^l k_j)T(r, g) \leq l(T(r, g) + T(r, f)) + Z(r, f') + S_f(r)$. Therefore

$$\begin{aligned} (\sum_{j=1}^l k_j)(T(r, f) + T(r, g)) &\leq 2l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r) \\ &\leq (2l + 1)(T(r, f) + T(r, g)) + S_f(r). \end{aligned}$$

So, $\sum_{j=1}^l k_j \leq 2l + 1$. Thus, since $\sum_{j=1}^l k_j > 2l + 1$ we have $u = 1$.

And if $\alpha \in \mathbb{K}$, we obtain

$$\begin{aligned} \sum_{j=1}^l k_j (T(r, f) + T(r, g)) &\leq 2l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r) \\ &\leq (2l + 1)(T(r, f) + T(r, g)) - 3 \log r + O(1) \end{aligned}$$

because $T(r, f') \leq T(r, f) - \log r + O(1)$, hence $\sum_{j=1}^l k_j \leq 2l$ which also contradicts the hypothesis

$c \neq 1$ whenever $\sum_{j=1}^l k_j > 2l$

Consequently, in the general case, whenever $\sum_{j=1}^l k_j > 2l + 1$, we have $u = 1$ and therefore $f'P'(f) = g'P'(g)$ hence $P(f) - P(g)$ is a constant D . But then by Lemma 5, we have $P(f) = P(g)$. And since P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(0, R^-))$), we can conclude $f = g$. Similarly, if $f, g \in \mathcal{A}(\mathbb{K})$ and if α is a non-zero constant, we have $u = 1$ whenever $\sum_{j=1}^l k_j > 2l$ and we conclude in the same way. $\square$

On $\mathbb{K}$, we have this theorem from results of [4]:

Theorem A: Let $P, Q \in \mathbb{K}[x]$ satisfy one of the following two statements:

$$\begin{aligned} \sum_{a_i \in F'} k_i &\geq s - m + 2 \text{ (resp. } \sum_{a_i \in \Delta} k_i \geq s - m + 3,) \\ \sum_{b_j \in F''} q_j &\geq 2 \text{ (resp. } \sum_{b_i \in \Lambda} q_j \geq 3,) \end{aligned}$$

If two meromorphic functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = Q(g(x))$, $x \in \mathbb{K}$, (resp. $x \in d(a, R^-)$) then both f and g are constant (resp. belong to $\mathcal{M}_b(d(a, R^-))$).

and on $\mathbb{C}$, we have this theorem from results of [3]:

Theorem B: Let $P, Q \in \mathbb{C}[X]$ satisfy one of the two following two conditions:

$$\begin{aligned} \sum_{a_i \in F'} k_i &\geq s - m + 3, \\ \sum_{b_j \in F''} q_i &\geq 3, \end{aligned}$$

and if the polynomial $P(X) - Q(Y)$ has no factor of degree 1, then there is no non-constant function $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) - Q(g(x)) = 0 \forall x \in \mathbb{C}$.

From Theorem B we can derive the following Theorem C:

Theorem C: Let $P, Q \in \mathbb{C}[X]$ satisfy one of the two following conditions:

$$\sum_{a_i \in F'} k_i \geq s - m + 3.$$

$$\sum_{b_j \in F''} q_j \geq 3.$$

Then there is no non-constant function $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) - Q(g(x)) = 0 \forall x \in \mathbb{C}$.

Proof of Theorem C: Let $F(X, Y) = P(X) - Q(Y)$. Since $\mathbb{C}$ is algebraically isomorphic to an ultrametric field such as $\mathbb{C}_p$ (with p any prime integer), without loss of generality we can transfer the problem onto the field $\mathbb{C}_p$. So, the image of the polynomial F in $\mathbb{C}_p[X, Y]$ is a polynomial $\tilde{F}(X, Y)$.

Thus, the hypothesis $\sum_{a_i \in F'} k_i \geq s - m + 3$ still holds in $\mathbb{C}_p$ and similarly, for the hypothesis

$$\sum_{b_j \in F''} q_j \geq 3. \text{ Suppose for instance } \sum_{a_i \in F'} k_i \geq s - m + 3. \text{ By Theorem B, there is no pair of non-}$$

constant functions $f, g \in \mathcal{M}(\mathbb{C}_p)$ such that $P(f(x)) - Q(g(x)) = 0$. Particularly, $\tilde{F}(X, Y)$ admits no factor of degree 1 in $\mathbb{C}_p[X, Y]$. But then, $F(X, Y)$ does not admit a factor of degree 1 in $\mathbb{C}[X, Y]$ either, because the factorization is conserved by a transfer. Now, we can apply Theorem B proving that when two functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = Q(g(x)) \forall x \in \mathbb{C}$, then they are constant.

Proof of Proposition P1: Suppose that two functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{K}$, $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(a, R^-)$)). We can apply Theorem A by putting $Q(X) = P(X) + C$. So, we have $h = l$ and $b_i = a_i$, $i = 1, \dots, l$. Let Γ be the curve of equation $P(X) - P(Y) = C$. By hypothesis we have $n \geq 2$, hence $\deg(P) \geq 3$, so Γ is of degree ≥ 3 . Therefore, if Γ has no singular point, it is of genus ≥ 1 and hence, by Picard-Berkovich Theorem, the conclusion is immediate. Consequently, we can assume that Γ has a singular point (α, β) . But then $P'(\alpha) = P'(\beta) = 0$ and hence (α, β) is of the form (a_h, a_k) . Consequently, $C = P(a_h) - P(a_k)$ and since $C \neq 0$, we have $h \neq k$. We will prove that either $a_1 \in F'$, or $a_1 \in F''$.

Suppose first that $a_1 \notin F' \cup F''$. Since $a_1 \notin F'$, there exists $i \in \{2, \dots, l\}$ such that $P(a_1) = P(a_i) + C$. Now since $1 \notin F''$, there exists $j \in \{2, \dots, l\}$ such that $P(a_1) + C = P(a_j)$. But since $C = -P(a_i)$, we have $P(a_j) = -P(a_i)$, therefore $P(a_i) + P(a_j) = 0$. Since P satisfies (G), we have $i = j$, hence $P(a_i) = 0$. But then $C = 0$, a contradiction. Therefore, we have proven that $a_1 \in F' \cup F''$. Now, by Theorem A, f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a, R^-))$).

Proof of Proposition P2: Suppose that two functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{C}$), $\forall x \in \mathbb{C}$. We will apply Theorem C by putting $Q(X) = P(X) + C$. Since $n \geq 3$, we have $\deg(P) \geq 4$ and hence Γ is of degree ≥ 4 . Consequently, if Γ has no singular point, it has genus ≥ 2 and hence, by Picard's Theorem, there exists no functions $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) = P(g(x)) + C$, $x \in \mathbb{C}$. Consequently, we can assume that Γ admits a singular point (a_h, a_k) . The proof is then similar to that of Proposition 1.

Notation: Let $f \in \mathcal{M}(\mathbb{C})$ be such that $f(0) \neq 0, \infty$. We denote by $Z_{[2]}(r, f)$ the counting function of the zeros of f each being counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

The following basic lemma applies to both complex and meromorphic functions. A proof is given in [5] for p -adic meromorphic functions and in [6] for complex meromorphic functions.

The following Theorem Y is indispensable in the proof of theorems:

Theorem Y: Let $P(x) = (x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{E}[x]$ ($a_i \neq a_j$, $\forall i \neq j$) with $l \geq 2$ and $n \geq \max\{k_2, \dots, k_l\}$ and let $k = \sum_{i=2}^l k_i$. Let $f, g \in \mathcal{M}(\mathbb{E})$ be transcendental (resp. let $f, g \in \mathcal{M}(d(a, R^-))$) and let $\theta = P(f)f'P(g)g'$. If θ belongs to $\mathcal{M}_f(\mathbb{E}) \cap \mathcal{M}_g(\mathbb{E})$, (resp. if θ belongs to $\mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) then we have the following :

if $l = 2$ then n belongs to $\{k, k+1, 2k, 2k+1, 3k+1\}$,
 if $l = 3$ then n belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2-k, 3k_3-k\}$,
 if $l \geq 4$ then $n = k+1$.
 Moreover, if f, g belong to $\mathcal{M}(\mathbb{K})$ and if θ is a constant, then $n = k+1$.
 Further, if f, g belong to $\mathcal{A}(\mathbb{E})$, then θ does not belong to $\mathcal{A}_f(\mathbb{E})$.

Lemma 6. Let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d'0, R^-)$), resp. $f \in \mathcal{M}(\mathbb{C})$). Then

$$T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + O(1).$$

Now, we can extract the following Lemma 7 from a result that is proven in several papers and particularly in Lemma 3 [14] when $\mathbb{E} = \mathbb{C}$ and, with precisions in Lemma 11 [5] when $\mathbb{E} = \mathbb{K}$.

Lemma 7. Let $f, g \in \mathcal{M}(\mathbb{C})$ (resp. $f, g \in \mathcal{M}(\mathbb{K})$) share the value 1 C.M. If $\Psi_{f,g}$ is not identically zero, then, $\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) + S_f(r) + S_g(r)$ (resp. $\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) - 6 \log r$).

We will need the following Lemma 8:

Lemma 8. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-)$), resp. $f, g \in \mathcal{M}(\mathbb{C})$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq \deg(Q) + 2$ (resp. $n \geq \deg(Q) + 3$, resp. $n \geq \deg(Q) + 3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.

For simplicity, we can assume $a_1 = 0$. Set $F = \frac{f'P'(f)}{\alpha}$ and $G = \frac{g'P'(g)}{\alpha}$. Clearly F and G share the value 1 C.M. Since f, g are transcendental, we notice that so are F and G . We put

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$$

We will prove that under the hypotheses of Theorems, $\Psi_{F,G}$ is identically zero.

The following lemma holds in the same way in p -adic analysis and in complex analysis. It is proven in [5] for the p -adic version and in [21] for the complex version. :

Lemma 9. Let $f, g \in \mathcal{M}(\mathbb{E})$ (resp. let $f, g \in \mathcal{M}(d(0, R^-)$) be non-constant and sharing the value 1 C.M. Suppose that $\Psi_{f,g} = 0$ and that

$$\limsup_{r \rightarrow +\infty} \left(\frac{\overline{Z}(r, f) + \overline{Z}(r, g) + \overline{N}(r, f) + \overline{N}(r, g)}{\max(T(r, f), T(r, g))} \right) < 1$$

(resp.

$$\limsup_{r \rightarrow R^-} \left(\frac{\overline{Z}(r, f) + \overline{Z}(r, g) + \overline{N}(r, f) + \overline{N}(r, g)}{\max(T(r, f), T(r, g))} \right) < 1.)$$

Then either $f = g$ or $fg = 1$.

Proofs of Theorem 2, 3, 4, 5, 6, 7

For simplicity, now we set $n = k_1$. Set $F = \frac{f'P'(f)}{\alpha}$, $G = \frac{g'P'(g)}{\alpha}$ and $\widehat{F} = P(f)$, $\widehat{G} = P(g)$. Suppose $F \neq G$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in K[x]$ of degree k . Now, with help of Lemma 6, we can check that we have

$$T(r, \widehat{F}) - Z(r, \widehat{F}) \leq T(r, \widehat{F}') - Z(r, \widehat{F}') + O(1)$$

Consequently, since $(\widehat{F})' = \alpha F$, we have

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, \widehat{F}) - Z(r, F) + T(r, \alpha) + O(1), \quad (1)$$

hence, by (1), we obtain

$$T(r, \widehat{F}) \leq T(r, F) + (n+1)Z(r, f) + Z(r, Q(f)) - nZ(r, f) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1).$$

i.e.

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1). \quad (2)$$

and similarly,

$$T(r, \widehat{G}) \leq T(r, G) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + T(r, \alpha) + O(1). \quad (3)$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \quad (4)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + O(1) \quad (5)$$

And particularly, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \quad (6)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + O(1). \quad (7)$$

Suppose now that $\Psi_{F,G}$ is not identically zero.

Let us place us in the p -adic context: $\mathbb{E} = \mathbb{K}$. By Lemma 7, we have

$$T(r, F) \leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) - 3 \log r$$

hence by (2), we obtain

$$\begin{aligned} T(r, \widehat{F}) &\leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) + Z(r, f) + Z(r, Q(f)) \\ &\quad - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) - 3 \log r + O(1) \end{aligned}$$

and hence by (4) and (5):

$$\begin{aligned} T(r, \widehat{F}) &\leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + \\ &\quad 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1) \quad (8) \end{aligned}$$

and similarly,

$$\begin{aligned} T(r, \widehat{G}) &\leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) \\ &\quad + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3 \log r + O(1). \quad (9) \end{aligned}$$

Consequently,

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (Z(r, f') + Z(r, g')) + \\ &\quad 4(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 6 \log r + O(1). \quad (10) \end{aligned}$$

Moreover, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then by (6) and (7) we have

$$\begin{aligned} T(r, \widehat{F}) &\leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + \\ &\quad Z(r, g') + 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1) \end{aligned}$$

and similarly,

$$T(r, \widehat{G}) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) +$$

$$Z(r, f') + 2\overline{N}(r, f) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3\log r + O(1).$$

Consequently,

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + Z(r, Q(f)) + Z(r, Q(g)) + \\ &\quad (Z(r, f') + Z(r, g')) + 4(\overline{N}(r, f) + \overline{N}(r, g)) + 4T(r, \alpha) - 6\log r + O(1) \end{aligned} \quad (11)$$

Now, let us go back to the general case. By Lemma 4 we can write $Z(r, f') + Z(r, g') \leq Z(r, f - a_2) + Z(r, g - a_2) + \overline{N}(r, f) + \overline{N}(r, g) - 2\log r$. Hence, in general, by (10) we obtain

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(Z(r, f) + Z(r, g)) &+ \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 8\log r + O(1) \end{aligned}$$

and hence, since $T(r, Q(f)) = kT(r, f) + O(1)$ and $T(r, Q(g)) = kT(r, g) + O(1)$,

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(T(r, f) + T(r, g)) &+ \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8\log r + O(1). \end{aligned} \quad (12)$$

Since $\widehat{F}$ is a polynomial in f of degree $n + k + 1$, we have $T(r, \widehat{F}) = (n + k + 1)T(r, f) + O(1)$ and similarly, $T(r, \widehat{G}) = (n + k + 1)T(r, g) + O(1)$, hence by (12) we can derive

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq \\ 5(T(r, f) + T(r, g)) &+ (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8\log r + O(1). \end{aligned} \quad (13)$$

hence

$$\begin{aligned} (n + k + 1)(Tr, f) + T(r, g) &\leq \\ 10(T(r, f) + T(r, g)) &+ \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) \\ &+ k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8\log r + O(1) \end{aligned}$$

hence

$$n(Tr, f) + T(r, g) \leq 9(T(r, f) + T(r, g)) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2)))$$

$$(14) \quad + \sum_{i=3}^l (4 - k_i) ((Z(r, f - a_i) + Z(r, g - a_i))) + 4T(r, \alpha) - 8 \log r + O(1))$$

Then $(5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \leq \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1)$ and at least, for each $i = 3, \dots, l$ we have $(4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \leq \max(0, 4 - k_i)(T(r, f) + T(r, g)) + O(1)$.

Now suppose $s_5 > 0$. That means that $k_i \geq 5 \forall i = 3, \dots, u_5$ with $l \geq 5$. We notice that the number of indices i superior or equal to 2 such that $k_i \geq 5$ is $u_5 - 2$. Similarly, for each $m > 5$, the number of indices superior or equal to 1 such that $k_i \geq m$ is $u_m - 1$.

Suppose first $\mathbb{E} = \mathbb{K}$. then we can apply Theorem N3 and then we obtain $\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) - \log r + O(1)$ and for each $m \geq 6$, $\sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) - \log r + O(1)$, i.e. $\sum_{i=3}^{u_5} Z(r, f - a_i) \geq s_5 T(r, f) - \log r + O(1)$ i.e. $\sum_{i=3}^{u_m} Z(r, g - a_i) \geq s_m T(r, g) - \log r + O(1)$ in Theorems 2, 3, 5.

Consequently, by (14) we obtain

$$(15) \quad n(T(r, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l \max(0, 4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) - \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1))$$

therefore

$$n \leq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{j=5}^{\infty} s_j,$$

a contradiction to the hypotheses of Theorems 2.

Consider now the situation in Theorems 4 and 5. In Theorem 4, we have $T(r, \alpha) \leq \log r + O(1)$ and in Theorem 5, $T(r, \alpha) = 0$. Consequently, Relation (15) now implies

$$n(T(r, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l \max(0, 4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) - \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) - 4 \log r + O(1))$$

therefore

$$n < 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=5}^{\infty} s_m$$

but this is incompatible with the hypothesis

$$n \geq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{j=5}^{\infty} s_j).$$

Now, let us consider the complex context: $\mathbb{E} = \mathbb{C}$. All inequalities above hold just by replacing each expression $-q \log r$ by $S_f(r) + S_g(r)$. However, we cannot apply Theorem N3 here but only Theorem N1. Therefore we obtain

$$\sum_{i=3}^{u_5} (Z(r, f - a_i) + Z(r, g - a_i)) \geq (u_5 - 4)(T(r, f) + T(r, g)) = t_5(T(r, f) + T(r, g))$$

and

$$\sum_{i=3}^{U_m} (Z(r, f - a_i) + Z(r, g - a_i)) \geq (u_m - 3)(T(r, f) + T(r, g)) = t_m(T(r, f) + T(r, g)).$$

Therefore we obtain

$$n \leq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=5}^{\infty} t_m$$

a contradiction to the hypothesis of Theorem 3.

Finally, consider the situation in Theorem 6. Since $N(r, f) = N(r, g) = 0$, Relation (13) gets

$$(n + k + 1)(T(r, f) + T(r, g)) \leq 5(T(r, f) + T(r, g)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) + S_f(r) + S_g(r).$$

On the other hand, by applying Theorem N1 to f and g , which now are entire functions, we have

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) = s_5 T(r, f), \quad \sum_{i=3}^{u_5} Z(r, g - a_i) \geq (u_5 - 3)T(r, g) = s_5 T(r, g)$$

and

$$\sum_{i=3}^{u_m} Z(r, f - a_i) \geq (u_m - 2)T(r, f) = s_m T(r, f), \quad \sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) = s_m T(r, g).$$

Consequently, $n + k + 1 \leq 5 + k + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=1}^{\infty} s_m$ and therefore

$n \leq 4 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=1}^{\infty} s_m$, a contradiction to the hypotheses of Theorem 6. Thus, in the hypotheses of Theorems 2, 3, 4, 5, 6 we have proven that $\Psi_{F,G}$ is

identically zero. Henceforth, we can assume that $\Psi_{F,G} = 0$ in each theorem. Note that we can

write $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = \left(\frac{F'}{(F-1)^2} \right) \left(\frac{(G-1)^2}{G'} \right)$. Since $\Psi_{F,G} = 0$, there exist $A, B \in \mathbb{E}$ such that

$$(16) \quad \frac{1}{G-1} = \frac{A}{F-1} + B$$

and $A \neq 0$.

We notice that $\overline{Z}(r, f) \leq T(r, f)$,

$$\overline{N}(r, f) \leq T(r, f) \overline{Z}(r, f - a_i) \leq T(r, f - a_i) \leq T(r, f) + O(1), \quad i = 2, \dots, l$$

and $\overline{Z}(r, f') \leq T(r, f') \leq 2T(r, f) + O(1)$. Similarly for g and g' . Moreover, if $\mathbb{E} = \mathbb{K}$ by Lemma 4 we have

$$(17) \quad T(r, F) \geq (n+k)T(r, f).$$

and if $\mathbb{E} = \mathbb{C}$, we have

$$(18) \quad T(r, F) \geq (n+k)T(r, f) - m(r, \frac{1}{f'}) + S_f(r).$$

We will show that $F = G$ in each theorem. We first notice that according to all hypotheses in Theorems 2, 3, 4, 5, 6 we have

$$(19) \quad n+k \geq 2l+7$$

and in Theorem 6, we have

$$(20) \quad n+k \geq 2l+5.$$

We will consider the following two cases: $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

Suppose $A \neq 1$. Then, by (16), we have $F = AG + (1-A)$. Suppose first $\mathbb{E} = \mathbb{K}$. Applying Theorem N1 to F , we obtain

$$(21) \quad \begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}\left(r, F - (1-A)\right) + \overline{N}(r, F) - \log r + O(1) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) \\ &\quad + \overline{Z}(r, f') + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) - \log r + O(1). \end{aligned}$$

By (17) and (21) we obtain

$$(22) \quad \begin{aligned} (n+k)T(r, f) &\leq \overline{Z}(r, F) + \overline{Z}\left(r, F - (1-A)\right) + \overline{N}(r, F) - \log r + O(1) \\ &\leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) - \log r + O(1). \end{aligned}$$

By (22) we have

$$\begin{aligned} (n+k)T(r, f) &\leq \overline{Z}(r, F) + \overline{Z}\left(r, F - (1-A)\right) + \overline{N}(r, F) - \log r + O(1) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) \\ &\quad + \overline{Z}(r, f') + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) - \log r + O(1). \end{aligned}$$

hence

$$(n+k)T(r, f) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i)$$

$$(23) \quad +\overline{N}(r, f) + \overline{Z}(r, g') + \overline{Z}(r, f') - \log r + O(1).$$

Then, considering all the previous inequalities, by Lemma 4 we can derive the following from (23):

$$(24) \quad (n+k)T(r, f) \leq (l+3)T(r, f) + (l+2)T(r, g) - 3 \log r + O(1).$$

Since f and g satisfy the same hypothesis, we also have

$$(25) \quad (n+k)T(r, g) \leq (l+3)T(r, g) + (l+2)T(r, f) - 3 \log r + O(1).$$

Hence, adding (24) and (25), we have

$$(n+k)[T(r, f) + T(r, g)] \leq (2l+5)[T(r, f) + T(r, g)] - 6 \log r + O(1)$$

therefore

$$(26) \quad n+k < 2l+5.$$

A contradiction to (20) proving that $A \neq 1$ is impossible whenever $B = 0$, in Theorems 2, 3, 5.

Suppose now $\mathbb{E} = \mathbb{C}$. By (18) we have

$$(n+k)T(r, f) \leq \overline{Z}(r, F) + \overline{Z}\left(r, F - \left(1-A\right)\right) + \overline{N}(r, F) + m\left(r, \frac{1}{f'}\right) + S_F(r) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) \\ + \overline{Z}(r, f') + m\left(r, \frac{1}{f'}\right) + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + S_f(r) + S_g(r).$$

Here we notice that $\overline{Z}(r, f') + m\left(r, \frac{1}{f'}\right) \leq T(r, \frac{1}{f'}) = T(r, f') + O(1)$, hence $(n+k)T(r, f) \leq$

$$(27) \quad \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{N}(r, f) + \overline{Z}(r, g') + T(r, f') + S_f(r) + S_g(r).$$

Then, considering all the previous inequalities in (27), similarly we can derive

$$(28) \quad (n+k)T(r, f) \leq (l+3)T(r, f) + (l+2)T(r, g) + S_f(r) + S_g(r).$$

Since f and g satisfy the same hypothesis, we also have

$$(29) \quad (n+k)T(r, g) \leq (l+3)T(r, g) + (l+2)T(r, f) + S_f(r) + S_g(r).$$

Hence, adding (28) and (29), we have

$$(n+k)[T(r, f) + T(r, g)] \leq (2l+5)[T(r, f) + T(r, g)] + S_f(r) + S_g(r)$$

therefore $n+k \leq 2l+5$, a contradiction to (20) proving that $A \neq 1$ is impossible whenever $B = 0$, in Theorem 3.

Consider now the situation in Theorem 6. By hypothesis we have $k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m)$ hence $n+k \geq 10 + 4(l-2) - \sum_{m=5}^{\infty} s_m = 4l+2 - \sum_{m=5}^{\infty} s_m$

Since $N(r, f) = N(r, g) = 0$, we can use Theorem N1 for entire functions and we obtain

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) + S_f(r) + S_g(r)$$

and for each $m \geq 6$,

$$\sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) + S_f(r) + S_g(r)$$

i.e.

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq s_5 T(r, f) + S_f(r) + S_g(r)$$

and

$$\sum_{i=3}^{u_m} Z(r, g - a_i) \geq s_m T(r, g) + S_f(r) + S_g(r).$$

Now, Relation (13) now gets

$$(n + k + 1)(T(r, f) + T(r, g)) \leq 5(T(r, f) + T(r, g)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2))$$

$$+ \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + k(T(r, f) + T(r, g)) + S_f(r) + S_g(r)$$

therefore $n + k \leq 9 + 4(l - 2) - \sum_{j=5}^{\infty} s_j = 2l + 1 - \sum_{m=5}^{\infty} s_m$ a contradiction to the hypothesis $n + k \geq 2l + 5$ of Theorem 6. Consequently, the hypothesis $A \neq 1$ does not hold when $B = 0$.

Henceforth we suppose $B \neq 0$.

Case 2: $B \neq 0$.

Consider first the situation when $\mathbb{E} = \mathbb{K}$, i.e. in Theorems 2 and in Theorems 4 and 5. By (17) we have Immediately,

$$\begin{aligned} \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) &\leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') \\ &\quad + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + \overline{N}(r, g) + 4T(r, \alpha) + O(1) \\ &\leq (l+1)[T(r, f) + T(r, g)] + T(r, f') + T(r, g') + 4T(r, \alpha) + O(1) \leq (l+3)(T(r, f) + T(r, g)) + 4T(r, \alpha) - 2 \log r \end{aligned}$$

hence by Lemma 4,

$$(30) \quad \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq (l+3)(T(r, f) + T(r, g)) + 4T(r, \alpha) - 2 \log r + O(1)$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and by Lemma 4, we have

$$T(r, f) \leq \frac{1}{n+k} (T(r, F) + T(r, \alpha)) + O(1)$$

and $T(r, g) \leq \frac{1}{n+k}(T(r, G) + T(r, \alpha)) + O(1)$. Consequently,

$$T(r, f) + T(r, g) \leq 2 \left[\frac{1}{n+k}(T(r, F) + T(r, \alpha)) \right] + O(1).$$

$$(31) \quad \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{n+k}T(r, F) + \left(\frac{2l+6}{n+k} + 4\right)T(r, \alpha) - 2\log r + O(1).$$

Now, by Hypotheses, in Theorems 2, 3, 5 by (19) we have $n+k \geq 2l+7$. Consequently, by Relation (31) we obtain

$$(32) \quad \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, F) + \left(\frac{2l+6}{2l+7} + 4\right)T(r, \alpha) + O(1).$$

and similiary

$$(33) \quad \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, G) + \left(\frac{2l+6}{2l+7} + 4\right)T(r, \alpha) + O(1).$$

hence

$$\limsup_{r \rightarrow +\infty} \left(\frac{\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G)}{\max(T(r, F), T(r, G))} \right) < 1$$

Therefore, by Lemma 9, in Theorems 2, 3, 5 we have either $F = G$, or $FG = 1$.

Suppose $FG = 1$. Then $f'P'(f)g'P'(g) = \alpha^2$. But in Theorems 2, 3, 5 we have assumed that $n \neq k+1$ and if $l = 2$, then $n \neq 2k, 2k+1, 3k+1$ and if $l = 3$ then $n \neq k, 3k_2 - k, 3k_3 - k$. Consequently, we have a contradiction to Theorem Y. Thus, the hypothesis $FG = 1$ is impossible and therefore we have $F = G$.

Consider now the situation when $\mathbb{E} = \mathbb{C}$, i.e. in Theorems 3 and 6. The proof is very similar to that in the case when $\mathbb{E} = \mathbb{K}$. We have

$$\overline{Z}(r, F) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + S_f(r)$$

and

$$\overline{N}(r, F) \leq \overline{N}(r, f) + S_f(r)$$

and similarly for G , so we can derive

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + \overline{Z}(r, g)$$

$$+ \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + \overline{N}(r, g) + S_f(r) + S_g(r)$$

$$(34) \quad \leq (l+2)[T(r, f) + T(r, g)] + S_f(r) + S_g(r).$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 4, we have

$$T(r, f) \leq \frac{1}{n+k} T(r, F) + S_f(r)$$

$$\text{and } T(r, g) \leq \frac{1}{n+k} T(r, G) + S_g(r). \text{ Consequently,}$$

$$T(r, f) + T(r, g) \leq \frac{2}{n+k} T(r, F) + S_f(r) + S_g(r).$$

Thus, (34) implies

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{n+k} T(r, F) + S_f(r) + S_g(r).$$

Now, as in Theorems 2, 3, 5, we can check that $n+k \geq 2l+7$ in theorem 3. Consequently, the previous inequality implies

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{2l+7} T(r, F) + S_f(r) + S_g(r)$$

and similarly,

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{2l+6}{2l+7} T(r, G) + S_f(r) + S_g(r)$$

hence by Lemma 9 again, we have $F = G$ or $FG = 1$. Then, by Theorem Y as in Theorems 2, 3, 5, the hypotheses of Theorem 3 prevent the case $FG = 1$ and therefore $F = G$.

Consider now the situation in Theorem 6. Relation (34) implies

$$(35) \quad \overline{Z}(r, F) + \overline{Z}(r, G) \leq (l+2)[T(r, f) + T(r, g)] + S_f(r) + S_g(r).$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 4, we have

$$T(r, f) \leq \frac{1}{n+k} T(r, F) + S_f(r) \text{ and } T(r, g) \leq \frac{1}{n+k} T(r, G) + S_g(r). \text{ Consequently,}$$

$$T(r, f) + T(r, g) \leq \frac{2}{n+k} T(r, F) + S_f(r) + S_g(r).$$

Thus, (220) implies

$$\begin{aligned} \overline{Z}(r, F) + \overline{Z}(r, G) &\leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + \overline{Z}(r, g) \\ &\quad + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + S_f(r) + S_g(r) \\ &\leq 4[T(r, f) + T(r, g)] + S_f(r) + S_g(r). \end{aligned}$$

Therefore

$$\overline{Z}(r, F) + \overline{Z}(r, G) \leq \frac{2l+4}{n+k} T(r, F) + S_f(r) + S_g(r)$$

hence by (20) we have

$$\overline{Z}(r, F) + \overline{Z}(r, G) \leq \frac{2l+4}{2l+5}T(r, F) + S_f(r) + S_g(r).$$

In the same way, this proves that either $F = G$ or $FG = 1$. But by Theorem Y, $FG = 1$ is impossible. Hence $F = G$.

Thus, in Theorems 2, 3, 4, 5, 6 we have proven that $F = G$ i.e. $f'P'(f) = g'P'(g)$. Consequently, $P(f) - P(g)$ is a constant C . Then, by Lemma 8 and Proposition P1, in Theorems 2, 3, 5 we have $P(f) = P(g)$ and by Lemma 8 and Proposition P2, we have $P(f) = P(g)$ in Theorems 3 and 6. Finally, in each theorem, P is a polynomial of uniqueness for the family of functions we consider. Consequently, $f = g$.

References

- [1] **An, T.T.H., Wang J.T.Y. and Wong, P.M.** *Unique range sets and uniqueness polynomials in positive characteristic II.* Acta Arithmetica, p. 115-143 (2005).
- [2] **An, T.T.H., Wang J.T.Y. and Wong, P.M.** *Strong uniqueness polynomials: the complex case.* Complex Variables, Vol. 49, No. 1, p. 25-54 (2004).
- [3] **T.T.H. An and N. T. N. Diep** *Genus one factors of curves dened by separated variable polynomials.* Journal of Number Theory, 133, 2616-2634 (2013).
- [4] **An, T. T. H. and Escassut, A** *Meromorphic solutions of equations over non-Archimedean fields.* Ramanujan Journal Vol. 15, No. 3, p.415-433 (2008)
- [5] **Boussaf, K. , Escassut, A. and Ojeda, J.** *p -adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function.* Bulletin des Sciences Mathématiques 136, n.2, p.172-200 (2012).
- [6] **Boussaf, K. , Escassut, A. and Ojeda, J.** *Complex meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function.* Indagationes (N.S.) 24, no. 1, 15-41(2013).
- [7] **Boutabaa, A.** *Théorie de Nevanlinna p -adique.* Manuscripta Math, 67, p. 251-269 (1990).
- [8] **Escassut, A.** *Analytic Elements in p -adic Analysis.* World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [9] **Escassut, A.** *Meromorphic functions of uniqueness,* Bulletin des Sciences Mathématiques, 131(3), p. 219-241 (2007).
- [10] **Escassut, A.** *p -adic Value Distribution.* Some Topics on Value Distribution and Differentiability in Complex and P -adic Analysis, p. 42- 138. Mathematics Monograph, Series 11. Science Press.(Beijing 2008).
- [11] **Fujimoto, H.** *On uniqueness of Meromorphic Functions sharing finite sets,* Amer. J. Math. 122 no. 6, p. 1175-1203 (2000).
- [12] **Hayman W. K.,** *Meromorphic Functions.* Oxford University Press, (1975)
- [13] **Hu, P.C. and Yang, C.C.** *Meromorphic Functions over non-Archimedean Fields.* Kluwer Academic Publishers, (2000).

- [14] **Hua, X. and Yang, C.C.** *Uniqueness and value-sharing of meromorphic functions*. Ann. Acad. Sci. Fenn. Math. 22, p. 395-406 (1997).
- [15] **Lin, W. and Yi, H.** *Uniqueness theorems for meromorphic functions concerning fixed-points*. Complex Var. Theory Appl., 49(11), p. 793-806 (2004).
- [16] **Nevanlinna, R.** *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthiers-Villars, Paris (1929).
- [17] **Ojeda, J.** *Uniqueness for ultrametric analytic functions*. Bulletin Mathématique des Sciences Mathématiques de Roumanie 54 (102, n.2, p.153-165 (2011).
- [18] **Wang, J.T.Y** *Uniqueness polynomials and bi-unique range sets*. Acta Arithmetica. 104, p. 183-200. (2002).
- [19] **Yang C.C. and Li, P.** *Some further results on the functional equation $P(f) = Q(g)$* . Series Advances in complex analysis and applications, Value distribution theory and related topics, p. 219-231, Kluwer Academic Publishers, Dordrecht-Boston-London (2004).
- [20] **Yang C.C. and Hua, X.** *Unique polynomials of entire and meromorphic functions*. Matematicheskaya Fizika Analiz Geometriye, v. 4, n.3, p. 391-398, (1997).
- [21] **Yi, H.X.** *Meromorphic functions that share one or two values*. Complex variables and applications. 28, p. 1-11. (1995).

Alain ESCASSUT
 Laboratoire de Mathématiques UMR 6620
 Université Blaise Pascal
 Les Cézeaux
 63171 AUBIERE
 FRANCE
 alain.escassut@math.univ-bpclermont.fr

Kamal BOUSSAF
 Laboratoire de Mathématiques UMR 6620
 Université Blaise Pascal
 Les Cézeaux
 63171 AUBIERE
 FRANCE
 kamal.boussaf@math.univ-bpclermont.fr

Jacqueline OJEDA
 Departamento de Matematica
 Facultad de Ciencias Fisicas y Matematicas
 Universidad de Concepcion
 CONCEPCION
 CHILE
 jacqojeda@udec.cl