p-adic Nevanlinna Theory outside of a hole

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Abstract

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field of characteristic 0, take $R > 0$ and let D be the set $\{x \in \mathbb{K} \mid |x| \geq R\}$. Let $\mathcal{M}(D)$ be the field of meromorphic functions in D. We contruct a Nevanlinna Theory for $\mathcal{M}(D)$ and show many properties similar to those previously obtained with meromorphic functions in $\mathbb{K}$ or in an open disk. Functions have at most one Picard value and at most four branched values. The functions with finitely many poles in D have no Picard value and at most one branched value. URSCM and URSIM are similar to those in complex analysis. Fujimoto's way lets obtain polynomials of uniqueness for $\mathcal{M}(D)$ with a degree ≥ 5. Many algebraic curves admit no parametrization by functions of $\mathcal{M}(D)$. Motzkin Factors, known for analytic elements, here play an essential role.

1 Introduction

In [12], M.O. Hanyak and A.A. Kondratyuk constructed a Nevanlinna theory for meromorphic functions in a *punctured complex plane*, i.e. in the set of the form $\mathbb{C} \setminus \{a_1, ..., a_m\}$, where we understand that the meromorphic functions can admit essential singularities at $a_1, ..., a_m$ [12]. Here we consider the situation in a complete p-adic algebraically closed field $\mathbb{K}$ of characteristic 0. The construction of a p-adic Nevanlinna theory was examined by Ha Huy Khoai [11] and A. Boutabaa [2], in the whole field and next a similar theory was made for unbounded meromorphic functions in an "open" disk of $\mathbb{K}$ [3].

Here we mean to construct a Nevanlinna theory for meromorphic functions in the complement of an open disk thanks to the use of specific properties of the Analytic Elements on infraconnected subsets of $\mathbb{K}$ [14], [8] and particularly the Motzkin Factorization [15] (see also [1] and chapter 32 in [6]). We can also obtain a Nevanlinna theory on 3 small functions [13], [8], as it was done in the classical context. Once the Nevanlinna Theory is established for such functions, we can apply it to obtain results on uniqueness and branched values as it was done in similar problems [3], [8], [13].

Notation: We denote by $\mathbb{K}$ a complete ultrametric algebraically closed field of characteristic 0 (such as $\mathbb{C}_p$). Given $r > 0$, $a \in \mathbb{K}$ we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$, by $d(a, r^-)$

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the disk $\{x \in \mathbb{K} \mid |x-a| < r\}$ and by $C(a,r)$ the circle $\{x \in \mathbb{K} \mid |x-a| = r\}$. Finally we fix $R > 0$ and denote by I the interval $[R, +\infty[$. Given $r'' > r'$, we put $\Delta(0, r', r'') = d(0, r'') \setminus d(0, r'^-)$. Throughout the paper, we denote by S the disk $d(0, R^-)$ and put $D = \mathbb{K} \setminus S$.

Given a bounded function f in D, we put $\|f\| = \sup_D |f(x)|$. Given a subset E of $\mathbb{K}$ having infinitely many points, we denote by $R(E)$ the $\mathbb{K}$-algebra of rational functions $h \in \mathbb{K}(x)$ having no pole in E. We then denote by $H(E)$ the $\mathbb{K}$-vector space of analytic elements on E [6] i.e. the completion of $R(E)$ with respect to the topology of uniform convergence on E. When E is unbounded, we denote by $H_0(E)$ the $\mathbb{K}$-subvector space of the $f \in H(E)$ such that $\lim\limits_{|x| \to +\infty} f(x) = 0$.

By classical properties of analytic elements [6], we know that given a circle $C(a,R)$ and an element f of $H(C(a,R))$ i.e. a Laurent series $f(x) = \sum\limits_{-\infty}^{+\infty} c_n (x-a)^n$ converging whenever $|x| = r$, then $|f(x)|$ is equal to $\sup\limits_{n \in \mathbb{Z}} |c_n| r^n$ in all classes of the circle $C(a,r)$ except maybe in finitely many. When $a = 0$, we put $|f|(r) = \sup\limits_{n \in \mathbb{Z}} |c_n| r^n$. Then $|f|(r)$ is a multiplicative norm on $H(C(0,r))$ (Chapters 13 and 19, Proposition 19.1, [8]).

We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$ and by $\mathcal{A}(D)$ the $\mathbb{K}$-algebra of Laurent series converging in D. Similarly, we will denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$ and by $\mathcal{M}(D)$ the field of fractions of $\mathcal{A}(D)$ that we will call *field of meromorphic functions in D*.

Given $f \in \mathcal{M}(D)$, for $r > R$, we will denote by $Z_R(r,f)$ the counting function of zeros of f between R and r, i.e. if $\alpha_1, ..., \alpha_m$ are the distinct zeros of f in $\Delta(0,R,r)$, with respective multiplicity u_j, $1 \le j \le m$, then $Z_R(r,f) = \sum\limits_{j=1}^{m} u_j(\log(r) - \log(|\alpha_j|))$. Similarly, we denote by $N_R(r,f)$ the counting function of poles of f between R and r, i.e. if $\beta_1, ..., \beta_n$ are the distinct poles of f in $\Delta(0,R,r)$, with respective multiplicity v_j, $1 \le j \le m$, then $N_R(r,f) = \sum\limits_{j=1}^{n} v_j(\log(r) - \log(|\beta_j|))$. Finally we put $T_R(r,f) = \max\big(Z_R(r,f), N_R(r,f)\big)$.

Next, we denote by $\overline{Z}_R(r,f)$ the counting function of zeros without counting multiplicity: if $\alpha_1, ..., \alpha_m$ are the distinct zeros of f in $\Delta(0,R,r)$, then we put

$\overline{Z}_R(r,f) = \sum\limits_{j=1}^{m} \log(r) - \log(|\alpha_j|)$.

Similarly, we denote by $\overline{N}_R(r,f)$ the counting function of poles without counting multiplicity: if $\beta_1, ..., \beta_n$ are the distinct poles of f in $\Delta(0,R,r)$, then we put

$\overline{N}_R(r,f) = \sum\limits_{j=1}^{n} \log(r) - \log(|\beta_j|)$.

Finally, putting $W = \{a_1, ..., a_q\}$, we denote by $Z_R^W(r,f')$ the counting function of zeros of f' on points where $f(x) \notin W$.

Throughout the paper, we denote by $|\ .\ |_\infty$ the Archimedean absolute value of $\mathbb{R}$. Given two functions defined in an interval $I = [b, +\infty[$, we will write $\phi(r) = \psi(r) + O(\log(r))$ (resp. $\phi(r) \le \psi(r) + O(\log(r))$) if there exists a constant $B > 0$ such that $|\phi(r) - \psi(r)|_\infty \le B\log(r)$, $r \in I$ (resp. $\phi(r) - \psi(r) \le B\log(r)$, $r \in I$).

We will write $\phi(r) = o(\psi(r))$, $r \in I$ if $\lim\limits_{r \to +\infty} \dfrac{\phi(r)}{\psi(r)} = 0$.

According to classical properties of analytic elements on infraconnected sets, we can recall the following lemmas that we will implicitely use [6], [8]:

Lemma L.I.1: *Let $f \in \mathcal{M}(D)$. If f has infinitely many zeros (resp. infinitely many poles) in D, the set of zeros (resp. the set of poles) is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} |\alpha_n| = +\infty$. If f has no zero in D, then it is of the form $\sum_{-\infty}^{+\infty} a_n x^n$ with $|a_q|r^q > |a_n|r^n \ \forall n \in \mathbb{Z},\ n \neq q, \forall r \geq R$.*

Lemma L.I.2: *Let $f,\ g \in \mathcal{A}(D)$. If $|f|(r) > |g|(r)$, then $|f+g|(r) = |f|(r)$.*

Definition: Let $f \in H(D)$ have no zero in D, $f(x) = \sum_{-\infty}^{+\infty} a_n x^n$ with $|a_q|r^q > |a_n|r^n \ \forall n \in \mathbb{Z},\ n \neq q,\ \forall r \geq R$. If $a_q = 1$, f will be called *a Motzkin factor associated to* S and the integer q will be called *the Motzkin index of* f and will be denoted by $m(f,S)$ [1], [15].

Theorem T.I.1: *Let $f \in \mathcal{M}(D)$. We can write f in a unique way, in the form $f^S f^0$ with $f^S \in H(D)$ a Motzkin factor associated to S and $f^0 \in \mathcal{M}(\mathbb{K})$, having no zero and no pole in S.*

Proof of Theorem T.I.1: Suppose first $f \in \mathcal{A}(D)$ and take $V > R$. Then as a quasi-invertible element of $H(\Delta(0,R,V))$ [6], [8], by Theorem 31.16 in [8], f admits a factorization in the form $f^S f^0$ where f^S is a Motzkin factor and f^0 belongs to $H(d(0,V))$ and has no zero in S. Moreover by Lemma 31.4 in [8], f^S does not depend on V. Consequently, since f^S is obviously invertible in $\mathcal{A}(D)$, we can factorize $f \in \mathcal{A}(D)$ in the form $f^S f^0$ where f^0 belongs to $\mathcal{A}(\mathbb{K})$ and has no zero in S.

Consider now the general case: $f = \dfrac{g}{h}$ with $g,\ h \in \mathcal{A}(D)$. Then we can write $g = g^S g^0,\ h = h^S h^0$, hence $f = \Big(\dfrac{g^S}{h^S}\Big)\Big(\dfrac{g^0}{h^0}\Big)$. Then we can check that this is the factorization announced in the statement: $f^S = \dfrac{g^S}{h^S}$ and $f^0 = \dfrac{g^0}{h^0}$. The uniqueness of this form is shown, for instance by Theorem 31.4 in [8].

The following Lemma LI.3 is immediate:

Lemma L.I.3: *The set of Motzkin factors associated to S makes a multiplicative group. Let $f,\ g \in \mathcal{M}(D)$. Then $(fg)^S = (f^S)(g^S)$, $\Big(\dfrac{1}{f}\Big)^S = \dfrac{1}{f^S}$, $(fg)^0 = (f^0)(g^0)$, $\Big(\dfrac{1}{f}\Big)^0 = \dfrac{1}{f^0}$ and $m(fg,S) = m(f,S) + m(g,S)$, $m(\dfrac{1}{f},S) = -m(f,S)$.*

Definitions and notations: We will denote by $\mathcal{M}^*(D)$ the set of $f \in \mathcal{M}(D)$ the set of f admitting at least infinitely many zeros in D or infinitely many poles in D. Similarly, we will denote by $\mathcal{A}^*(D)$ the set of $f \in \mathcal{A}(D)$ the set of f admitting infinitely many zeros in D. Next, we set $\mathcal{M}^0(D) = \mathcal{M}(D) \setminus \mathcal{M}^*(D)$ and $\mathcal{A}^0(D) = \mathcal{A}(D) \setminus \mathcal{A}^*(D)$.

Remark: *$\mathcal{M}^0(D)$ is a subfield of $\mathcal{M}(D)$.*

2 Nevanlinna Theory in $\mathcal{M}(D)$.

T.II.1 is similar to Corollary 22.27 in [8]:

Theorem T.II.1: *Let $f \in \mathcal{M}(D)$. Then* $\log(|f|(r)) - \log(|f|(R)) = Z_R(r,f) - N_R(r,f) + m(f,S)(\log r - \log R)$ *$(r \in I)$.*

Corollary T.II.1.1: *Let $f \in \mathcal{M}(D)$. Then $T_R(r,f)$ is identically zero if and only if f is a Motzkin factor.*

Corollary T.II.1.2: *Let $f \in \mathcal{A}(D)$ and let $\phi \in H_0(D)$. Then $Z_R(r, f+\phi) \leq Z_R(r,f) + O(\log(r))$ $(r \in I)$.*

Corollary T.II.1.3: *Let $f \in \mathcal{A}(D)$. Then $Z_R(r,f') \leq Z_R(r,f) + O(\log(r))$ $(r \in I)$.*

Corollary T.II.1.4: *Let $f,\ g \in \mathcal{A}(D)$ satisfy $\log(|f|(r)) \leq \log(|g|(r))$ $\forall r \geq R$ $(r \in I)$. Then $Z_R(r,f) \leq Z_R(r,g) + (m(g,S) - m(f,S))(\log(r) - \log(R))$, $(r \in I)$.*

Proof of Theorem T.II.1: By Theorem T.I.1, we have $f = f^S f^0$. Since f^S has no zero and no pole in D, by Lemma L.I.1 it satisfies $|f^S|(r)) = r^{m(f,S)}$ $\forall r \in I$, hence $\log(|f^S|(r)) - \log(|f^S|(R)) = m(f,S)(\log r - \log R)$ $(r \in I)$. Next, since f^0 has no zero and no pole in S, we have $\log(|f^0|(r)) - \log(|f^0|(R)) = Z_R(r,f^0) - N_R(r,f^0)$ $(r \in I)$ therefore the statement is clear.

We can now characterize the set $\mathcal{M}^*(D)$:

Theorem T.II.2: *Let $f \in \mathcal{M}(D)$. The three following statements are equivalent:*

i) $\displaystyle\lim_{r\to+\infty} \frac{T_R(r,f)}{\log(r)} = +\infty$ $(r \in I)$,

ii) $\dfrac{T_R(r,f)}{\log(r)}$ *is unbounded,*

iii) f *belongs to* $\mathcal{M}^*(D)$.

Proof of Theorem T.II.2: Consider an increasing sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that $\lim_{n\to+\infty} u_n = +\infty$ and let $(k_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{N}^*$. Clearly, we have

$$\lim_{r\to+\infty} \frac{\sum_{u_n \leq r} k_n(\log(r) - \log(u_n))}{\log(r)} = +\infty.$$

Consequently, if a function $f \in \mathcal{M}^*(D)$ has infinitely many zeros (resp. infinitely many poles in D) then $\displaystyle\lim_{n\to+\infty} \frac{Z_R(r,f)}{\log(r)} = +\infty$ (resp. $\displaystyle\lim_{n\to+\infty} \frac{N_R(r,f)}{\log(r)} = +\infty$) hence in both cases, $\displaystyle\lim_{n\to+\infty} \frac{T_R(r,f)}{\log(r)} = +\infty$. Conversely, if f has finitely many zeros and finitely many poles in D, then we check that

$$\lim_{n\to+\infty} \frac{T_R(r,f)}{\log(r)} < +\infty.$$

Thus the equivalence of the three statements is clear.

Operations on $\mathcal{M}(D)$ work almost like for meromorphic functions in the whole field [2], [8], so we will not develop the proof.

Theorem T.II.3: *Let $f,\ g \in \mathcal{M}(D)$. Then for $r \in I$, and for every $b \in \mathbb{K}$, $T_R(r, f+b) = T_R(r,f) + O(\log(r))$, $T_R(r, f.g) \leq T_R(r,f) + T_R(r,g) + O(\log(r))$, $T_R(r, \frac{1}{f}) = T_R(r,f))$, $T_R(r, f+g) \leq T_R(r,f) + T_R(r,g) + O(\log(r))$ and $T_R(r,f^n) = nT_R(r,f)$.*
Let h be a Moebius function. Then $T_R(r, h\circ f) = T_R(r,f) + O(\log(r))$. Moreover, if both f and g belong to $\mathcal{A}(D)$, then $T_R(r, f+g) \leq \max(T_R(r,f), T_R(r,g)) + O(\log(r))$ and $T_R(r,fg) = T_R(r,f) + T_R(r,g)$. Particularly, if $f \in \mathcal{A}^*(D)$, then $T_R(r,f+b) = T_R(r,f) + O(1)$.

Corollary T.II.3.1: *Let* $f,\ g \in \mathcal{M}^0(D)$*. Then* $T_R(r, \frac{f}{g}) \geq T_R(r,f) - T_R(r,g)\ (r \in I)$ *and* $T_R(r, \frac{f}{g}) \geq T_R(r,g) - T_R(r,f)\ (r \in I)$*.*

Theorem II.4 is easily proven thanks to properties of the Nevanlinna characteristic functions.

Theorem T.II.4: *Every* $f \in \mathcal{M}^*(D)$ *is transcendental over* $\mathcal{M}^0(D)$*.*

The proof of Theorem T.II.5 is uneasy in all context we can consider. Here we can follow a similar way as in [2], [8], taking into account expressions of the form $O(\log(r))$.

Theorem T.II.5: *Let* $f \in \mathcal{M}(D)$ *and let* $P(x) \in \mathbb{K}[x]$ *be of degree* q*. Then* $T_R(r, P \circ f) = qT_R(r,f) + O(\log(r))\ (r \in I)$*.*

Theorem II.6 is similar to what is known for meromorphic functions in the whole field or inside a disk, this is why we will avoid the proof.

Theorem T.II.6: *Let* $f \in \mathcal{M}(D)$*. Then* $N_R(r, f^{(k)}) = N_R(r,f) + k\overline{N}_R(r,f) + O(1),\ (r \in I)$ *and* $Z_R(r, f^{(k)}) \leq Z_R(r,f) + k\overline{N}_R(r,f) + O(\log(r)),\ (r \in I)$*.*

We can now examine some properties of the functions in $\mathcal{M}^*(D)$ and first, consider small functions.

Definitions and notation: For each $f \in \mathcal{M}(D)$ we denote by $\mathcal{M}_f(D)$ the set of functions $h \in \mathcal{M}(D)$ such that $T_R(r,h) = o(T_R(r,f))\ r \in I$. Similarly, if $f \in \mathcal{A}(D)$ we will denote by $\mathcal{A}_f(D)$ the set $\mathcal{M}_f(D) \cap \mathcal{A}(D)$.

The elements of $\mathcal{M}_f(D)$ are called *small meromorphic functions with respect to* f*, small functions* in brief. Similarly, if $f \in \mathcal{A}(D)$ the elements of $\mathcal{A}_f(D)$ are called *small analytic functions with respect to* f*, small functions in brief.*

A value $b \in \mathbb{K}$ will be called *a quasi-exceptional value for a function* $f \in \mathcal{M}(D)$ if $f - b$ has finitely many zeros. In the same way, a small function w with respect to a function $f \in \mathcal{M}(D)$ will be called *a quasi-exceptional small function for* f if $f - w$ has finitely many zeros in D.

Theorem T.II.7 is a direct and immediate application of Theorem T.II.3:

Theorem T.II.7: $\mathcal{A}_f(D)$ *is a* $\mathbb{K}$*-subalgebra of* $\mathcal{A}(D)$*,* $\mathcal{M}_f(D)$ *is a subfield of* $\mathcal{M}(D)$*.*

Let $f \in \mathcal{M}(D)$ *and let* $g \in \mathcal{M}_f(D)$*. Then* $T_R(r, fg) = T_R(r,f) + o(T_R(r,f))\ (r \in I)$ *and* $T_R(r, \frac{f}{g}) = T_R(r,f) + o(T_R(r,f))\ (r \in I)$*.*

Let $g,\ h \in \mathcal{A}(D)$ *with* g *and* h *not identically zero. If* gh *belongs to* $\mathcal{A}_f(D)$ *then so do* g *and* h*.*

Theorem T.II.8: *Let* $f \in \mathcal{M}^*(D)$*. There exists at most one function* $w \in \mathcal{M}_f(D)$*, such that* $f - w$ *have finitely many zeros in* D*. Moreover, if* f *has finitely many poles, then there exists no function* $w \in \mathcal{M}_f(D)$*, such that* $f - w$ *have finitely many zeros in* D*.*

Corollary T.II.8.1: *Let* $f \in \mathcal{M}^*(D)$*. Then* f *admits at most one quasi-exceptional small function. Moreover, if* f *belongs to* $\mathcal{A}^*(D)$*, then* f *has no quasi-exceptional small function.*

Proof of Theorem T.II.8: Suppose that f admits two distinct quasi-exceptional small functions. Without loss of generality we may assume that these functions are 0 and $w \in \mathcal{M}_f(D)$. Let

$g = f - w$, let f^S be the Motzkin factor of f associated to S and let g^S be the Motzkin factor of g associated to S. Then f is of the form $f^S \dfrac{P}{h}$ with $h \in \mathcal{A}^*(D)$ and $P \in \mathbb{K}[x]$ having all its zeros in D and g is of the form $g^S \dfrac{Q}{l}$ with $l \in \mathcal{A}^*(D)$ and $Q \in \mathbb{K}[x]$ having all its zeros in D. Consequently, we have $hg^S Q - lf^S P = hlw$. Now, by Theorem T.II.3, we have

$$T_R(r, hg^S Q - lf^S P) \leq \max(T_R(r, hg^S Q), T_R(r, lf^S P)) + O(\log(r)),$$

$$\leq (T_R(r, h), T_R(r, l) + O(\log(r)),\ (r \in I),$$

hence

$$T_R(r, hlw) \leq \max((T_R(r, h), T_R(r, l)) + O(\log(r)),\ (r \in I). \tag{1}$$

Next, by Theorem T.II.3 and Corollary T.II.3.1 we have

$$T_R(r, wlh) \geq T_R(r, h) + T_R(r, l) - T_R(r, w). \tag{2}$$

Now, since $f = f^S \dfrac{P}{h}$, clearly $T_R(r, f) = T_R(r, h) + O(\log(r))$ and similarly, $T_R(r, g) = T_R(r, l) + O(\log(r))$. But since $T_R(r, w) = o(T_R(r, f)$, we can check that $T_R(r, g) = T_R(r, f) + o(T_R(r, f))$. Consequently, $T_R(r, l) = T_R(r, h) + o(T_R(r, h))$. Therefore by (1) we have $T_R(r, hlw) \leq (T_R(r, h) + o(T_R(r, h),\ (r \in I)$ and by (2) we obtain $T_R(r, wlh) \geq 2T_R(r, h) + o(T_R(r, f))\ (r \in I)$, a contradiction. This proves that f cannot have two small functions $w \in \mathcal{M}_f(D)$ such that $f - w$ have finitely many zeros.

Suppose now that $f \in \mathcal{A}^*(D)$ has finitely many poles and admits a quasi-exceptional small function w. Set $g = f - w$. Then g is of the form $g^S \left(\dfrac{P}{h}\right)$ where P is a polynomial whose zeros lie in D and h belongs to $\mathcal{A}(D)$ and it admits for zeros the poles of f and those of w. Consequently, w belongs to $\mathcal{A}_f(D)$. Therefore we can check that $T_R(r, g) = o(T_R(r, f)$. But by Corollary T.II.3.1 we have $T_R(r, g) \geq T_R(r, f) - T_R(r, w) = T_R(r, f) + o(T_R(r, f))$, a contradiction.

The Nevanlinna second Main Theorem is based on the following theorem:

Theorem T.II.9: *Let $f \in \mathcal{M}(D)$ and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T_R(r, f) \leq \max_{1 \leq k \leq q} \Big(\sum_{j=1, j \neq k}^{q} Z_R(r, f - a_j) \Big) + O(\log(r))\ (r \in I).$$

Corollary T.II.9.1: *Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, ..., a_q \in \mathbb{K}$ be distinct. Then* $(q-1)T_R(r, f) \leq \sum_{j=1}^{q} Z_R(r, f - a_j) + O(\log(r))\ (r \in I)$.

Based upon Theorem T.II.9 we can give a proof of Theorem T.II.10 and T.II.11 in an easy way, as in classical case.

Theorem T.II.10: *Let $f \in \mathcal{M}(D)$, let $\alpha_1, ..., \alpha_q \in \mathbb{K}$, with $q \geq 2$ and let $W = \{\alpha_1, ..., \alpha_q\}$. Then*

$$(q-1)T_R(r, f) \leq \sum_{j=1}^{q} \overline{Z}_R(r, f - \alpha_j) + Z_R(r, f') - Z_R^W(r, f') + O(\log(r)) \quad (r \in I).$$

Moreover, if f belongs to $\mathcal{A}(D)$ then

$$qT_R(r,f) \le \sum_{j=1}^{q} \overline{Z}_R(r,f-\alpha_j) + Z_R(r,f') - Z_R^W(r,f') + O(\log(r)) \quad (r \in I).$$

Theorem T.II.11 (Second Main Theorem): *Let $f \in \mathcal{M}(D)$, let $\alpha_1, ..., \alpha_q \in \mathbb{K}$, with $q \ge 2$ and let $W = \{\alpha_1, ..., \alpha_q\}$. Then*

$$(q-1)T_R(r,f) \le \sum_{j=1}^{q} \overline{Z}_R(r,f-\alpha_j) + \overline{N}_R(r,f) - Z_R^W(r,f') + O(\log(r)) \quad (r \in I).$$

As in the classical case, the general Main Theorem lets us write a Nevanlinna theorem on three small functions:

Theorem T.II.12: *Let $f \in \mathcal{M}^*(D)$ and let $w_1,\ w_2, w_3 \in \mathcal{M}_f(D)$ be pairwise distinct. Then $T_R(r,f) \le \sum_{j=1}^{3} \overline{Z}_R(r,f-w_j) + \sum_{j=1}^{3} T_R(r,w_j) + O(\log(r))$ $(r \in I)$.*

Corollary T.II.12.1: *Let $f \in \mathcal{M}^*(D)$ and let $w_1,\ w_2, w_3 \in \mathcal{M}^0(D)$ be pairwise distinct. Then $T_R(r,f) \le \sum_{j=1}^{3} \overline{Z}_R(r,f-w_j) + O(\log(r))$ $(r \in I)$.*

Theorem T.II.13 is an easy consequence of Theorem T.II.12:

Theorem T.II.13: *Let $f \in \mathcal{M}^*(D)$ and let $w_1,\ w_2 \in \mathcal{M}_f(D)$ be distinct. Then*

$$T_R(r,f) \le \overline{Z}_R(r,f-w_1) + \overline{Z}_R(r,f-w_2) + \overline{N}_R(r,f) + T_R(r,w_1) + T_R(r,w_2) + O(\log(r)) \ (r \in I).$$

Corollary T.II.13.1: *Let $f \in \mathcal{M}^*(D)$ and let $w_1,\ w_2 \in \mathcal{M}^0(D)$ be distinct. Then*

$$T_R(r,f) \le \overline{Z}_R(r,f-w_1) + \overline{Z}_R(r,f-w_2) + \overline{N}_R(r,f) + O(\log(r)) \ (r \in I).$$

Corollary T.II.13.2: *Let $f \in \mathcal{A}^*(D)$ and let $w_1,\ w_2 \in \mathcal{A}_f(D)$ be distinct. Then*

$$T_R(r,f) \le \overline{Z}_R(r,f-w_1) + \overline{Z}_R(r,f-w_2) + T_R(r,w_1) + T_R(r,w_2) + O(\log(r)) \ (r \in I).$$

Corollary T.II.13.3: *Let $f \in \mathcal{A}^*(D)$ and let $w_1,\ w_2 \in \mathcal{A}^0(D)$ be distinct. Then*

$$T_R(r,f) \le \overline{Z}_R(r,f-w_1) + \overline{Z}_R(r,f-w_2) + O(\log(r)) \ (r \in I).$$

3 Applications.

in a similar way as in classical cases [8], in D we can prove the following theorems T.III.1 and T.III.2:

Theorem T.III.1: *Let $a_1,\ a_2,\ a_3 \in \mathbb{K}$ ($a_i \neq a_j\ \forall i \neq j$) and let $f,\ g \in \mathcal{A}^*(D)$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1,\ 2,\ 3$). Then $f = g$.*

Theorem T.III.2: *Let $a_1,\ a_2,\ a_3,\ a_4,\ a_5 \in \mathbb{K}$ ($a_i \neq a_j\ \forall i \neq j$) and let $f,\ g \in \mathcal{M}^*(D)$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1,\ 2,\ 3,\ 4,\ 5$). Then $f = g$.*

We can now apply Theorem T.II.11 to obtain results concerning certain algebraic curves:

Theorem T.III.3: *Let Λ be a curve of equation $y^q = P(x)$ with $P \in \mathbb{K}[x]$ having at least two distinct zeros. If Λ admits a parametrization of the form $y = g(t),\ x = f(t)$ with $f,\ g \in \mathcal{M}(D)$ and if f (resp. g) belongs to $\mathcal{M}^0(D)$, then g (resp. f) also belongs to $\mathcal{M}^0(D)$.*

Proof of Theorem T.III.3: Let $a_1, ..., a_n$ be n distinct zeros of P. Suppose that two functions $f,\ g \in \mathcal{M}(D)$ satisfy the equation $g(t)^q = P(f(t))$. If f belongs to $\mathcal{M}^0(D)$, then by Theorem T.II.4, g also belongs to $\mathcal{M}^0(D)$. Conversely, if g belongs to $\mathcal{M}^0(D)$, then f satisfies $P(f) - g^q = 0$, hence by Theorem T.II.4 f belongs to $\mathcal{M}^0(D)$.

Theorem T.III.4: *Let Λ be a curve of equation $y^q = P(x)$ with $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1. If Λ admits a parametrization of the form $y = g(t),\ x = f(t)$ with $f,\ g \in \mathcal{M}^*(D)$ and $t \in D$, then $n \leq \dfrac{2q}{q-1}$. Moreover, if $\deg(P) = n$ and if n and q are relatively prime, then $n \leq \dfrac{q+1}{q-1}$.*

Proof of Theorem T.III.4: Let $a_1, ..., a_n$ be n distinct zeros of P. Suppose that two functions $f,\ g \in \mathcal{M}^*(D)$ satisfy the equation $g(t)^q = P(f(t))$. Let $\alpha \in D$ be a zero of $f - a_j$ of order s. It is a zero of order l of $g - a_j$, hence $lq = s$ therefore q divides s. Consequently, for each $j = 1, ..., n$, we have $\overline{Z}_R(r, f - a_j) \leq \dfrac{1}{q} Z_R(r, f - a_j)$, hence, by Theorems T.II.3 and T.II.11 we have

$$(1) \qquad (n-1)T_R(r, f) \leq \frac{n}{q} T_R(r, f) + \overline{N}_R(r, f) + O(\log(r)).$$

Since $f,\ g$ belong to $\mathcal{M}^*(D)$, that implies $n \leq \frac{2q}{q-1}$.

Suppose now that $\deg(P) = n$ and that n and q are relatively prime. Let β be a pole of f of order l in D. Since $\deg(P) = n$, all zeros of P are of order 1 and β is a pole of g^q of order ln. But since n and q are relatively prime, q must divide l. Consequently, by (1) now we have

$$(n-1)T_R(r, f) \leq \frac{n}{q} T_R(r, f) + \frac{1}{q} T_R(r, f) + O(\log(r))$$

hence $(n-1)T_R(r, f) \leq (\dfrac{n+1}{q}) T_R(r, f) + O(\log(r))$. And since $f,\ g$ belong to $\mathcal{M}^*(D)$, that implies $n \leq \frac{q+1}{q-1}$.

By Theorems T.III.3 and T.III.4, we can get these corollaries:

Corollary T.III.4.1: *Let Λ be a curve of equation $y^q = P(x)$ with $q \geq 2$, $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1. If Λ admits a parametrization of the form $y = g(t)$, $x = f(t)$ with $f,\ g \in \mathcal{M}(D)$ and $t \in D$ and if $n > \dfrac{2q}{q-1}$ then the two both functions f and g belong to $\mathcal{M}^0(D)$.*

Corollary T.III.4.2: *Let Λ be a curve of equation $y^q = P(x)$ with $q \geq 2$ relatively prime to n and $P \in \mathbb{K}[x]$ of degree n admitting n distinct zeros. If Λ admits a parametrization of the form $y = g(t)$, $x = f(t)$ with $f,\ g \in \mathcal{M}(D)$ and if $n > \dfrac{q+1}{q-1}$, then the two both functions f and g belong to $\mathcal{M}^0(D)$.*

Example: Let Λ be a curve of equation $y^2 = P(x)$ with $\deg(P) = 5$, P admitting five distinct zeros. If two functions $f,\ g \in \mathcal{M}(D)$ satisfy $g(t)^2 = P(f(t))$, then the two both functions belong to $\mathcal{M}^0(D)$.

We can now consider the problem of branched rational functions.

Definition: Let $f \in \mathcal{M}^*(D)$ and let $w \in \mathbb{K}(x)$. Then w is called *a perfectly branched function, with respect to* f if all zeros of $f - w$ are multiple except maybe finitely many. Particularly, the definition applies to constants [4] .

Theorem T.III.5: *Let $f \in^* (D)$. Then f admits at most 4 perfectly branched values.*

Proof of Theorem T.III.5: Suppose f has q perfectly branched values b_j with $j = 1, ..., q$. For each j, let s_j be the number of simple zeros of $f - b_j$ and let $s = \sum_{j=1}^{q} s_j$. Applying Theorem T.II.11, we have

$$(q-1)T_R(r,f) \leq \sum_{j=1}^{q} \overline{Z}_R(r, f-b_j) + \overline{N}_R(r,f) + O(\log r). \tag{1}$$

But since $f - b_j$ has s_j simple zeros, we have

$$\overline{Z}_R(r, f-b_j) \leq \frac{Z_R(r, f-b_j) + s_j \log r}{2} + O(1) \leq \frac{T_R(r,f) + s_j \log r}{2} + O(1)\ \forall j = 1, ..., q$$

hence, by (1), we have

$$(q-1)T_R(r,f) \leq \frac{qT_R(r,f)}{2} + T_R(r,f) + O(\log(r)). \tag{2}$$

By (2) clearly we have $q \leq 4$ in all cases, which shows the statement of Theorem T.III.5 whenever $f \in \mathcal{M}(D)$.

Theorem T.III.6: *Let $f \in \mathcal{M}^*(D)$ have finitely many poles in D. Then f admits at most one perfectly branched function in $\mathcal{M}^0(D)$.*

Corollary T.III.6.1: *Let $f \in \mathcal{A}(D)$ have infinitely many zeros in D. Then f admits at most one perfectly branched function in $\mathcal{M}^0(D)$.*

In the proof of Theorem T.III.6, we will use the following lemma:

Lemma L.III.1. *Let* $\Theta(x)=\sum_{-\infty}^{+\infty}a_nx^n\in\mathcal{A}(D)$*, with* $a_0=1$*, have no zero in* D*. Take* $S>4R$ *and set* $D'=\mathbb{K}\setminus d(0,S^-)$ *Then there exists a function* $\sqrt{\Theta(x)}$ *defined in* D' *and belonging to* $\mathcal{A}(D')$*.*

Proof: Since Θ belongs to $\mathcal{A}(D)$ and has no zero in D, while $a_0=1$, we have $|a_n|R^n<1\ \forall n\in\mathbb{Z}$ and $\lim_{n\to+\infty,n\to-\infty}|a_n|R^n=0$. Then, it is well known that there exists a unique function $\ell\in\mathcal{A}(d(1,(\frac{R}{4})^-))$ with value in $d(1,1^-)$ such that $(\ell(u))^2=u\quad\forall u\in d(1,(\frac{R}{4})^-)$ (see for instance Theorem 31.23 in [8]). Here, we put $u=\sum_{-\infty}^{+\infty}a_nx^n$ and the function ℓ belongs to $\mathcal{A}(D')$ whenever $S>4R$.

Proof of Theorem T.III.6: Suppose that f admits two perfectly branched functions $w_1,\ w_2\in\mathcal{M}^0(D)$. If we consider the function $g=f-w_1$, we can see that g has two perfectly branched functions 0 and w_1-w_2 that both belong to $\mathcal{M}^0(D)$. So, without loss of generality, we may assume that f admits two perfectly branched functions 0 and $w(x)\neq 0$ which belong to $\mathcal{M}^0(D)$.

Suppose first that f has infinitely many zeros of order ≥ 3. Then $Z_R(r,f)-2\overline{Z}_R(r,f)$ is a function $\zeta(r)$ such that

(1) $$\lim_{r\to+\infty}\frac{\zeta(r)}{\log r}=+\infty.$$

and then
$$\overline{Z}_R(r,f)\leq\frac{T_R(r,f)-\zeta(r)}{2}.$$
On the other hand, by Corollary T.II.13.1, we have
$$T_R(r,f)\leq\overline{Z}_R(r,f)+\overline{Z}_R(r,f-w)+\overline{N}_R(r,f)+O(\log(r)).$$
Consequently by (1), we can derive

(2) $$T_R(r,f)\leq 2\overline{Z}_R(r,f-w)+2\overline{N}_R(r,f)-\zeta(r)+O(\log(r)).$$

Now, let q be the number of simple zeros of $f-w$ in D and let s be the number of distinct poles of f in D. Since $\overline{Z}(r,f-w)\leq\frac{T(r,f)}{2}+q\log(r)$, by (2) we can derive
$$T_R(r,f)\leq T_R(r,f)+2(q+s)\log(r)-\zeta(r)+O(\log(r))$$
hence $0\leq-\zeta(r)+O(\log(r))$. But by (1) we have a contradiction proving that f cannot admit 0 and w as branched functions lying in $\mathcal{M}^0(D)$.

Symmetrically if $f-w$ has infinitely many zeros of order ≥ 3, we can obviously conclude in the same way.

Therefore, we are now led to assume that all zeros of both f and $f-w$ are of order 2 except finitely many. Since f has infinitely many zeros in D and since $w\in\mathcal{M}^0(D)$, there exists $V>4R$ satisfying the following properties:
i) all poles of w in D lie in $\Delta(0,R,V)$
ii) $|f|(r)>|w|(r)\ \forall r>V$,
iii) all zeros of f and of $f-w$ in $\mathbb{K}\setminus\Delta(0,R,V)$ are of order 2 exactly.

Let $S' = d(0, V^-)$ and let $D' = \mathbb{K} \setminus S'$. Then f obviously belongs to $\mathcal{M}(D')$. Therefore, by Theorem T.I.1, f admits in $\mathcal{M}(D')$ a Motzkin factor of the form $x^s\theta$ with $s = m(f, S')$ and then we can write f in the form $x^s\theta g^2$ and $g \in \mathcal{A}(\mathbb{K})$, having no zero in S'. Similarly, $f - w$ admits a Motzkin factor of the form $x^t\tau$ with $t = m(f - w, S')$ and we can then write $f - w$ in the form $x^t\tau h^2$ and $h \in \mathcal{A}(\mathbb{K})$, having no zero in S'.

Since $|f|(r) > |w|(r)\ \forall r \geq V$, we can check that f and $f - w$ have the same number of zeros on each circle $C(0, r)$ $(r \geq V)$ and $f|(r) = |f - w|(r)\ \forall r \geq V$. Consequently, $s = t$. We have the equality

$$g^2 - \frac{\tau}{\theta}h^2 = \frac{w}{x^s\theta}. \tag{2}$$

Now, $\dfrac{\tau}{\theta}$ is a Motzkin factor of index zero, hence by Lemma L.III.1, it admits a square root $\Xi \in H(\mathbb{K} \setminus S')$ of the form $\sum_{-\infty}^{0} a_n x^n$ with $|a_n|V^n < |a_0|\ \forall n < 0$. Then by (2) we have

$$(g - \Xi h)(g + \Xi h) = \frac{w}{x^s\theta}. \tag{3}$$

We will check that this equality is impossible. Indeed, both functions $g - \Xi h$ and $g + \Xi h$ belong to $\mathcal{A}(D')$. Suppose $Z_V(r, g - \Xi h) = O(\log(r))$. Then $g + \Xi h = (g - \Xi h) + 2\Xi h$ satisfies $Z_V(r, g - \Xi h) + Z_V(r, \Xi h) = Z_V(r, \Xi h) + O(\log(r))$ and consequently,

$$\lim_{r\to+\infty}\Big[\frac{Z_V(r, (g - \Xi h)(g + \Xi h))}{\log(r)}\Big] = +\infty$$

because

$$\lim_{r\to+\infty}\Big[\frac{Z_V(r, \Xi h)}{\log(r)}\Big] = +\infty.$$

But on the other hand, by construction, $Z_V(r, \dfrac{w}{x^s\theta})$ is of the form $O(\log(r))$, which shows that (3) is impossible. That ends the proof of Theorem T.III.6.

Notation Given $f \in \mathcal{M}(D)$ and $a \in D$, we denote by $\omega_a(f)$ the order of f at a i.e. if f admits a as a zero of order q, we put $\omega_a(f) = q$, if f admits a as a pole of order q, we put $\omega_a(f) = -q$ and if a is neither a zero nor a pole of f, we put $\omega_a(f) = 0$.

Let $h \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$ (resp. $h \in E(x) \setminus E$) and let $\Lambda(h)$ be the set of zeros c of h' such that $h(c) \neq h(d)$ for every zero d of h' other than c. If $\Lambda(h)$ is finite, we denote by $\Upsilon(h)$ its cardinal and if $\Lambda(h)$ is not finite, we put $\Upsilon(h) = +\infty$.

Theorem T.III.7: *Let $P \in \mathbb{K}[x]$, let $f,\ g \in \mathcal{A}^*(D) \setminus \mathbb{K}$ satisfy $P \circ f = P \circ g$. If $\Upsilon(P) \geq 3$, then $f = g$.*

Theorem T.III.8: *Let $P \in \mathbb{K}[x]\ \mathbb{K}$, let $f,\ g \in \mathcal{M}^*(D) \setminus \mathbb{K}$ satisfying further $P \circ f = P \circ g$. If $\Upsilon(P) \geq 4$, then $f = g$.*

Examples: 1) Let $P(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - x^2$. Then $P'(x) = x(x - 1)(x + 2)$. Then $P(0) = 0,\ P(1) = \dfrac{1}{4} + \dfrac{1}{3} - 1,\ P(2) = \dfrac{8}{3}$.

Thus the three zeros a_j of P' satisfy $P(a_i) \neq P(a_j)\ \forall i \neq j$. Consequently, $\Upsilon(P) = 3$ and hence P is a polynomial of uniqueness for $\mathcal{A}^*(D)$.

2) Let $P(x) = \frac{x^5}{5} - \frac{5x^3}{3} + 4x$. Then $P'(x) = (x-1)(x+1)(x-2)(x+2)$. $P(1) = \frac{1}{5} - \frac{5}{3} + 4$, $P(-1) = -P(-1)$, $P(2) = \frac{32}{5} - \frac{24}{3} + 8 = -P(-2)$.

Thus the four zeros a_j of P' satisfy $P(a_i) \neq P(a_j)\ \forall i \neq j$. Consequently, $\Upsilon(P) = 4$ and hence P is a polynomial of uniqueness for $\mathcal{M}^*(D)$.

The hypothesis $\Upsilon(P) \geq 4$ however is not necessary to prove that a polynomial is a polynomial of uniqueness for $\mathcal{M}^*(D)$, as shows the following Theorem T.III.9. The proof is similar to that of Theorem 41.9 in [8] and first came from [9].

In the proof of Theorems T.III.7 and T.III.8 we will need the following lemma which is similar to Lemma 10 in [7] (see also Lemma 53.1 in [8]):

Lemma L.III.2: *Let $P(x) \in \mathbb{K}[x] \setminus \mathbb{K}$ and let $f,\ g \in \mathcal{M}(D)$ satisfy $P \circ f = P \circ g$. Let $W = \{c_1, ..., c_l\}$ be the set of zeros of P'.*

For each $j = 1, ..., k$ $(k \leq l)$ let $q_j = \omega_{c_j}(P')$. We assume that $P(c_j) \neq P(c_n)\ \forall j = 1, ..., k,\ \forall n \leq l$. Then $f,\ g$ satisfy

$$\overline{N}_R(r, f) + \sum_{j=1}^{k} \overline{Z}_R(r, f - c_j) \leq \overline{Z}_R(r, \frac{1}{f} - \frac{1}{g}) + \sum_{j=1}^{k} \frac{1}{q_j} Z_R(r, g' \mid f(x) = c_j,\ g(x) \notin W).$$

Furthermore, if $f,\ g \in \mathcal{A}(D)$, then

$$\sum_{j=1}^{k} \overline{Z}_R(r, f - c_j) \leq \overline{Z}_R(r, f - g) + \sum_{j=1}^{k} \frac{1}{q_j} Z_R(r, g' \mid f(x) = c_j,\ g(x) \notin W).$$

The following lemma is immdiate:

Lemma L.III.3: *Let $P \in \mathbb{K}[x]$ and let $f,\ g \in \mathcal{A}^*(D)$ satisfy $P \circ f = P \circ g$. Then $T_R(r, f) = T_R(r, g) + O(\log(r))$ $r > R)$.*

Proofs of Theorems T.III.7 and T.III.8: Suppose that f and g are not identical. Let W be the set of zeros of P' and let $c_1, ..., c_k$ lie in W. Clearly by applying Theorem T.II.11 we obtain respectively in Theorems T.III.7 and T.III.8

(1) $$(k-1)T_R(r, f) \leq \sum_{j=1}^{k} \overline{Z}_R(r, f - c_j) + \overline{N}_R(r, f) - Z_R^W(r, f') + O(\log(r)),\ (r > R),$$

(2) $$(k-1)T_R(r, g) \leq \sum_{j=1}^{k} \overline{Z}_R(r, g - c_j) + \overline{N}_R(r, g) - Z_R^W(r, g') + O(\log(r)),\ (r > R).$$

Now, let $\phi = \dfrac{1}{f} - \dfrac{1}{g}$ and for each $j = 1, ..., k$, let $q_j = \omega_{c_j}(P')$. By (1) and (2) and by Lemma L.III.2 we obtain

$$(k-1)T_R(r,f) \leq \overline{Z}_R(r,\phi) + \sum_{j=1}^{k} \frac{1}{q_j} Z_R(r, g' \mid f(x) = c_j,\ g(x) \notin W)$$

$$-Z_R(r, f' \mid f(x) \notin W + O(\log(r)) \tag{3}$$

and similarly:

$$(k-1)T_R(r,g) \leq \overline{Z}_R(r,\phi) + \sum_{j=1}^{k} \frac{1}{q_j} Z_R(r, f' \mid g(x) = c_j,\ f(x) \notin W)$$

$$-Z_R(r, g' \mid g(x) \notin W + O(\log(r)). \tag{4}$$

By adding in each case the two inequalities by (3) and (4) in Theorems T.III.7 and T.III.8, we obtain:

$$(k-1)(T_R(r,f) + T_R(r,g)) \leq 2\overline{Z}_R(r,\phi) +$$

$$+\sum_{j=1}^{k} \frac{1}{q_j} [Z_R(r, f' \mid g(x) = c_j,\ f(x) \notin S + Z_R(r, g' \mid f(x) = c_j,\ g(x) \notin W)]$$

$$-Z_R(r, f' \mid f(x) \notin W) - Z_R(r, g' \mid g(x) \notin W)] + O(\log(r)). \tag{5}$$

Now, in each inequality (5), we notice that in the left side member we have the term:

$$\sum_{j=1}^{k} \frac{1}{q_j} [Z_R(r, f' \mid g(x) = c_j,\ f(x) \notin S)] - Z_R(r, f' \mid f(x) \notin W)$$

which is clearly inferior or equal to zero and similarly

$$\sum_{j=1}^{k} \frac{1}{q_j} [Z_R(r, g' \mid f(x) = c_j,\ g(x) \notin S)] - Z_R(r, g' \mid g(x) \notin W) \leq 0.$$

Consequently, by Lemma L.III.2, in Theorems T.III.8 we obtain

$$(k-1)(T_R(r,f) + T_R(r,g)) \leq 2\overline{Z}_R(r,\phi) + O(\log(r))$$

and hence

$$(k-1)(T_R(r,f) + T_R(r,g)) \leq 2\overline{Z}_R(r, f-g) + O(\log(r)). \tag{6}$$

Now, by Theorem T.II.3 we have $Z_R(r,\phi) \leq T_R(r,f) + T_R(r,g) + O(\log(r))$ therefore $k \leq 3$ and hence, if $\Upsilon(P) \geq 4$, we have $f = g$.

Now assume the hypotheses of Theorem T.III.7. By Lemma L.III.2 we can replace $Z_R(r,\phi)$ by $Z_R(r,f-g)$. Next, by Lemma L.III.3, $T_R(r,f)=T_R(r,g)+O(\log(r))$, hence, by Theorem T.II.3, we can derive $T_R(r,f-g)\le T_R(r,f)+O(\log(r))=T_R(r,g)+O(\log(r))$. Consequently in place of (6), in Theorem T.III.7 we obtain

$$(k-1)(T_R(r,f)+T_R(r,g))\le 2\overline{Z}_R(r,f-g)+O(\log(r))\le T_R(r,f)+T_R(r,g)+O(\log(r)).$$

Thus we can conclude that $k\le 2$ in Theorem T.III.7 and hence, if $\Upsilon(P)\ge 3$, we have $f=g$.

Theorem T.III.9: *Let*

$$Q(x)=\Big((n+2)(n+1)x^{n+3}-2(n+3)(n+1)x^{n+2}+(n+3)(n+2)x^{n+1}\Big).$$

Then Q is a polynomial of uniqueness for $\mathcal{M}^*(D)$ for every $n\ge 3$.

The proof of Theorem T.III.9 is similar to this proposed for meromorphic functions in $\mathbb{K}$ or in $\mathbb{C}$ (see for instance [7], [8], [9]). It uses the following basic lemma L.III.4 (stated in [9]):

Lemma L.III.4: *Let E be an algebraically closed field of characteristic 0 and let*

$$P(x)=(n-1)^2(x^n-1)-n(n-2)(x^{n-1}-1)^2\in E[x].$$

Then P admits 1 as a zero of order 4 and all other zeros u_j $(1\le j\le 2n-6)$ are simple.

Notation: Following Theorem T.III.9, given $n\in\mathbb{N}$ and let $c\in\mathbb{K}$) we denote by $P_{n,c}$ the polynomial introduced in [10] and also used in [3] and [8]:

$$P_{n,c}(x)=(n-1)(n-2)x^n-2n(n-2)x^{n-1}+n(n-1)x^{n-2}+c$$

and by $L(n,c)$ be the set of zeros of $P_{n,c}$ in $\mathbb{K}$.

In order to state Theorem III.11, we need recall the notation $\mathcal{E}$ used with URSCM. Given a subset B of $\mathbb{K}$ and $f\in\mathcal{M}(D)$ we denote by $\mathcal{E}(f,B)$ the set in $\mathbb{K}\times\mathbb{N}^*$:
$\bigcup_{a\in B}\{(z,q)\in\mathbb{K}\times\mathbb{N}^*\mid z \text{ a zero of order } q \text{ of } f(x)-a\}$.

And given a subset B of $\mathbb{P}^1(\mathbb{K})$ containing $\{\infty\}$ and $f\in\mathcal{M}(D)$, we denote by $\mathcal{E}(f,B)$ the subset of $\mathbb{K}\times\mathbb{N}^*$: $\mathcal{E}(f,S\cap\mathbb{K})\cup\{(z,q)\ z \text{ a pole of order } q \text{ of } f\}$.

Theorem T.III.10: *Let $f,\ g\in\mathcal{M}^*(D)$ be two different non-constant functions satisfying $f^{-1}(L(n,c))=g^{-1}(L(n,c))$. Then $n\le 16$. Moreover, if $f,\ g\in\mathcal{A}^*(D)$, then $n\le 9$.*

Corollary T.III.10.1: *Let $n\ge 17$. Then $L(n,c)$ is an ursim for $\mathcal{M}^*(D)$. Let $n\ge 10$. Then $L(n,c)$ is an ursim for $\mathcal{A}^*(D)$.*

Theorem T.III.11: *Let $f,\ g\in\mathcal{M}^*(D)$ be two distinct non-constant functions satisfying $\mathcal{E}(f,L(n,c))=\mathcal{E}(g,L(n,c))$. Then $n\le 10$. Moreover, if $f,\ g\in\mathcal{A}^*(D)$ then $n\le 6$.*

Corollary T.III.11.1: *For every $n\ge 11$, $L(n,c)$ is an urscm for $\mathcal{M}^*(D)$ and for every $n\ge 7$, $L(n,c)$ is an urscm for $\mathcal{A}^*(D)$.*

Proof of Theorems T.III.10 and T.III.11: The proof of Theorems T.III.10 and T.III.11 are very close to the proofs of Theorems 1.6 and 1.11 in [3] and in [9] (see also Theorems 54.10 and

54.16 in [8]), just by replacing $O(1)$ by $O(\log(r))$ and also look like the proofs given in [9], thanks to Theorem T.II.11. The proofs use the following two lemmas L.III.5 and L.III.6.

Lemma L.III.5: *Let $F,\ G \in \mathcal{M}(D)$ have the same poles, ignoring multiplicitiy, and let $H = \frac{F''}{F'} - \frac{G''}{G'}$. Every pole of H has multiplicity order 1. Let α be a pole of F and G. If α has same multiplicity for F and G, then H has no pole at α. Moreover, if α has a multiplicity order 1 for both F and G, then α is a zero of H.*

Lemma L.III.6: *Let $f,\ g \in \mathcal{M}(D))$ be two different non-constant functions satisfying $\frac{f''}{f'} = \frac{g''}{g'}$. Then f and g are linked by a relation of the form $f = ag + b$.*

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