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Abstract Recently, methods have been proposed to reconstruct a 2-manifold surface from a sparse cloud of points estimated from an image sequence. Once a 3D Delaunay triangulation is computed from the points, the surface is searched by growing a set of tetrahedra whose boundary is maintained 2-manifold. Shelling is a step that adds one tetrahedron at once to the growing set. This paper surveys properties that helps to understand the shelling performances: shelling provides most tetrahedra enclosed by the final surface but it can "get stuck" or block in unexpected cases.

Keywords Reconstruction · Volumetric models · Shellability · 3D Delaunay Triangulation · Star-Shapes

1 Introduction

In the recent years, a family of 2-manifold surface reconstruction methods have been proposed to deal with a sparse cloud of input points and their visibility information estimated from an image sequence. There are both batch [13] and incremental [23, 15, 16, 18] variations. First a 3D Delaunay triangulation is computed from the sparse point cloud to divide up the space by a set T of tetrahedra. Then the visibility information is used to associate a free-space score to every tetrahedron $\Delta \in T$, e.g. by counting the rays (line segments between points and camera poses used to reconstruct these points) that intersect Δ. Last these scores are used for growing a set O of tetrahedra in T such that the boundary ∂O of O is maintained 2-manifold.

Several operations are designed for the O growing. One of them adds one tetrahedron at once to O and is closely related to the "shelling" process that is studied in Combinatorial Topology: Algorithm 1 in [13] generates a shelling in the 3D case if O is initialized by a single tetrahedron. This greedy algorithm tries to add to O the tetrahedron in $T \setminus O$ that has the largest visibility score (and that has a face triangle in ∂O) if ∂O is still a 2-manifold. The goal of the paper is to survey the shelling properties in our surface reconstruction context, where O is included in a 3D Delaunay triangulation and the O initialization is not restricted to a single tetrahedron.

First Sec. 2 introduces prerequisites. Second, Sec. 3 provides our shelling definition and shows that it does not change the topology (number of connected components and genus) of ∂O in almost all cases.

Third, Sec. 4 overviews shelling properties known in Combinatorial Topology, i.e. if O is initialized by a single tetrahedron. At first glance, one could guess that a shelling started by a single tetrahedron in T can reach/have every end value $O_e \subseteq T$ such that tetrahedron union $\cup O_e$ is homeomorphic to one tetrahedron. Surprisingly, this is wrong: some of these O_e do not have such a shelling. This implies that no greedy algorithm [13,18] (whatever the free-space scoring of the tetrahedra) can reach O_e. Furthermore, even if there is a shelling that reaches O_e, a greedy algorithm can "get stuck" or block: it can generate another shelling with other tetrahedron choice(s) at the beginning such that the end value is strictly included in O_e.

Fourth, Sec. 5 studies shelling in the favorable case of star-shapes in a 3D Delaunay triangulation. We remind that $\cup O$ is star-shaped with respect to the center $\mathbf{c} \in \mathbb{R}^3$, if for every point $\mathbf{x} \in \cup O$, the line segment $\mathbf{xc} \subseteq \cup O$. More precisely, we show that there is a shelling that starts from O_s and ends to $O_e \subseteq T$ if both $\cup O_s$ and $\cup O_e$ are star-shaped with respect to the same center. This generalizes a known case where $\cup O_e$ is convex (indeed, a convex set is star-shaped with respect to every of its point) and O_s has a single tetrahedron. Then Sec. 5 explains why shelling provides most of the tetrahedra enclosed by the final surface in our context (shelling does not have excessive blocking).

Last, Sec. 6 provides examples of shelling blocking: a shelling starts from O_s but cannot end to O_e although $\cup O_s$ and $\cup O_e$ have the same topology. This should convince the reader that such cases really exist, although they don't in the 2D case [4] (i.e. by replacing tetrahedra by triangles). The examples are chosen to be as simple as possible. Sec. 6 introduces a family of sets O that cannot be reached by a shelling started from a single tetrahedron (as in Sec. 4), which generalizes an example in [24]. It also introduces examples of visual artifact similar to that in Fig. 1 of [16] which cannot be removed by shelling alone (against the intuition).

2 Main prerequisites

The majority of prerequisites are in [5,21,9,11,14].

2.1 Simplicial complexes in $\mathbb{R}^n$

Let integers $k \geq 0$ and $n > 0$. A *simplex* σ is the convex hull of $k+1$ points $\mathbf{v}_0, \cdots, \mathbf{v}_k$ in general position in $\mathbb{R}^n$; σ is a *vertex* if $k = 0$, an *edge* if $k = 1$, a *triangle* if $k = 2$, a *tetrahedron* if $k = 3$. The *dimension* of σ is k. A simplex σ' is a *face* of σ if σ' is the convex hull of some of the $\mathbf{v}_i$ above (we have $\sigma' \subseteq \sigma$). A *simplicial complex* K in $\mathbb{R}^n$ is a finite set of simplices in $\mathbb{R}^n$ such that (1) $\sigma' \in K$ if σ' is a face of $\sigma \in K$ and (2) $\sigma \cap \sigma'$ is empty or a face of σ and σ' if both σ and σ' are in K.

A *3D Delaunay triangulation* is a simplicial complex K in $\mathbb{R}^3$ that meets several conditions. There is a set T of tetrahedra such that the faces of the tetrahedra are the simplices in K. Let V be the vertex set of K. The circumscribing sphere of every tetrahedron in T does not contain a vertex in V in its interior. The convex hull of V is the union of the tetrahedra in T.

2.2 Closure, pure simplicial complex, boundary

Let A be included in a simplicial complex. The *closure* $c(A)$ of A is the set of all faces (including the vertices) of the simplices in A. If both A and B are included in a same simplicial complex, $c(A \cup B) = c(A) \cup c(B)$, $c(A \cap B) \subseteq c(A) \cap c(B)$. If $A \subseteq B$, $c(A) \subseteq c(B)$. We use notation $c(\sigma_1, \cdots \sigma_k) = c(\{\sigma_1, \cdots \sigma_k\})$ for simplices σ_i.

Let X be a simplicial complex. We say that X is *kD pure* if there are simplices(s) $\sigma_i \in X$ with a same dimension k such that $X = c(\sigma_1, \cdots \sigma_m)$. A 3D Delaunay triangulation K is 3D pure and meets $K = c(T)$ where T is the set of all tetrahedra in K.

If A is a set of simplices with a same dimension $k > 0$, the *boundary* ∂A of A is the set of all simplices of dimension $k-1$ in $c(A)$ that are faces of exactly one simplex in A. The boundary $\partial\sigma$ of a simplex σ is $\partial\{\sigma\}$.

2.3 Abstract simplicial complexes

Let $\mathcal{V}$ be a finite set. An *abstract simplicial complex* $\mathcal{S}$ is a set of subsets of $\mathcal{V}$ such that $A \in \mathcal{S}$ and $B \subset A$ imply $B \in \mathcal{S}$. It is implicitly defined by every simplicial complex K as follows: $\mathcal{V}$ is the vertex set of K and $\mathcal{S}$ is the set of the vertex sets of the K simplices. Conversely, K is a *realization* of $\mathcal{S}$ in $\mathbb{R}^n$. The elements of $\mathcal{S}$ are called *(abstract) simplices*; their *faces* are their subsets.

The definitions of edge/triangle/tetrahedron, closure, kD pure and boundary still hold for abstract simplices and abstract simplicial complexes, replacing simplices by abstract simplices and using their same inclusion relations. Same notations are used for a simplicial complex and its abstract version, and also for a simplex and its abstract version. We use bold fonts for vertices, e.g. $\mathbf{ab}$ is an edge (abstract or non-abstract).

Let $c(T)$ be a 3D Delaunay triangulation using a non-empty set T of tetrahedra. Every triangle in $c(T)$ is included in exactly two tetrahedra in $c(T)$, except those in ∂T (we have $\partial T \neq \emptyset$). These exceptions are removed as in [2,14] to make easier proofs. Let $\mathbf{v}_\infty$ be an abstract vertex that is different to those in $c(T)$. For every triangle $\mathbf{abc} \in \partial T$, we create a new tetrahedron $\mathbf{abcv}_\infty$ by adding $\mathbf{v}_\infty$ to the set $\mathbf{abc}$. Let $T^\infty = T \cup \{\mathbf{abcv}_\infty, \mathbf{abc} \in \partial T\}$. Now $T \subset T^\infty$ and every triangle in $c(T^\infty)$ is a face of exactly 2 tetrahedra in T^∞. Notations T and T^∞ are used in the whole paper.

2.4 2-manifolds (with boundary), k-spheres, k-balls

Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. If there is a bijective and continuous function between A and B such that the inverse function is also continuous, A and B are *homeomorphic*. We say that A is a *2-manifold with boundary* if every point $\mathbf{x} \in A$ has a neighborhood in A that is homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}^+$ ($\mathbb{R}^+$ includes 0); A is a *2-manifold* if every point $\mathbf{x} \in A$ has a neighborhood in A that is homeomorphic to $\mathbb{R}^2$; A is a *k-ball* ($k \in \{1,2,3\}$) if it is homeomorphic to $\{\mathbf{x} \in \mathbb{R}^k, ||\mathbf{x}|| \leq 1\}$; A is a *k-sphere* ($k \in \{1,2\}$) if it is homeomorphic to $\{\mathbf{x} \in \mathbb{R}^{k+1}, ||\mathbf{x}|| = 1\}$. 2-spheres are 2-manifolds, 2-manifolds and 2-balls are 2-manifolds with boundary.

Let $Y \subseteq c(T)$. We define $|Y| = \cup Y = \cup_{\sigma \in Y} \sigma$ (here σ is a convex hull, it is not an abstract simplex). We say that Y is *homeomorphic* to B if $|Y|$ is homeomorphic to B. Thus Y can be a *2-manifold/2-manifold with boundary/k-sphere/k-ball.* Here are examples: a triangle in $c(T)$ is a 2-ball, ∂T is a 2-sphere, $\partial \Delta$ is a 2-sphere and Δ is a 3-ball if tetrahedron $\Delta \in c(T)$.

Let vertex series $\mathbf{v}_1 \cdots \mathbf{v}_m$ where $m \geq 2$. We use notation $\mathbf{v}_1$-$\mathbf{v}_2$-$\cdots$-$\mathbf{v}_m$ for the set of edges $\mathbf{v}_1\mathbf{v}_2, \mathbf{v}_2\mathbf{v}_3, \cdots, \mathbf{v}_{m-1}\mathbf{v}_m$. If the $\mathbf{v}_i$ are distinct and every $\mathbf{v}_i\mathbf{v}_{i+1} \in c(T)$, $\mathbf{v}_1$-$\mathbf{v}_2$-$\cdots$-$\mathbf{v}_m$ is a 1-ball. A *cycle* is an edge set $\mathbf{v}_1$-$\cdots$-$\mathbf{v}_m$-$\mathbf{v}_1$ such that $m \geq 3$ and the $\mathbf{v}_i$ are distinct. A cycle is a 1-sphere if all its edges are in $c(T)$.

Here we merge Theorem 4 and Lemma 4 in [14]:

Theorem 1 *Let $O \subset T^\infty$ be such that $\partial O \subset c(T)$ and ∂O is a 2-manifold. If Δ is a tetrahedron in $T \setminus O$, $\partial(O \cup \{\Delta\})$ is a 2-manifold iff $c(O) \cap c(\Delta)$ is 2D pure.*

2.5 2-manifold criterion for simplicial complexes

If $\mathbf{v}$ is a vertex and E is a set of edges, $\mathbf{v} \times E$ is the set of the triangles defined by joining $\mathbf{v}$ and every edge in E: $\mathbf{v} \times E = \{\mathbf{vab}, \mathbf{ab} \in E\}$. Let X be a set of triangles in $c(T)$. If a vertex $\mathbf{v} \in c(X)$, we define $X_\mathbf{v}$ by the set of all triangles in X that include $\mathbf{v}$. There is a set $E_\mathbf{v}$ of edges such that $X_\mathbf{v} = \mathbf{v} \times E_\mathbf{v}$. If $E_\mathbf{v}$ is not a cycle, $\mathbf{v}$ is a *singular vertex* of X.

According to [21,1,14], X is a 2-manifold iff for every vertex $\mathbf{v} \in c(X)$, there is a cycle $\mathbf{a}_*$ such that $X_\mathbf{v} = \mathbf{v} \times \mathbf{a}_*$. Thus there are distinct vertices $\mathbf{a}_i \in c(X_\mathbf{v})$ such that $\mathbf{a}_* = \mathbf{a}_1$-$\cdots$-$\mathbf{a}_m$-$\mathbf{a}_1$. This implies that every edge $\mathbf{vw} \in c(X)$ is included in exactly two triangles in X (indeed, $\mathbf{w}$ is an $\mathbf{a}_i$ and all $\mathbf{a}_i$ are distinct).

According to [5,9], X is a 2-manifold with boundary iff for every vertex $\mathbf{v} \in c(X)$, there is a non-empty set of edge(s) $\mathbf{a}_*$ such that $X_\mathbf{v} = \mathbf{v} \times \mathbf{a}_*$ with two cases: there are distinct vertices $\mathbf{a}_i$ such that $\mathbf{a}_* = \mathbf{a}_1$-$\cdots\mathbf{a}_m$ or $\mathbf{a}_* = \mathbf{a}_1$-$\cdots\mathbf{a}_m$-$\mathbf{a}_1$ is a cycle ($m \geq 2$ in the first case and $m \geq 3$ in the second case). Thus every $\mathbf{a}_i \in c(X_\mathbf{v})$ and every edge $\mathbf{vw} \in c(X)$ is included in one or two triangle(s) in X (but not more).

2.6 Connectivity and genus

Let $X \subseteq c(T^\infty)$. We say that X is *connected* if for all vertices $\mathbf{v}$ and $\mathbf{v}'$ in $c(X)$, there are vertices $\mathbf{v}_i \in c(X)$ such that $1 \leq i \leq m$, $\mathbf{v} = \mathbf{v}_1$, $\mathbf{v}' = \mathbf{v}_m$ and $\mathbf{v}_1$-$\mathbf{v}_2$-$\cdots$-$\mathbf{v}_m \subseteq c(X)$. A *connected component* X is a connected subset of X that is maximal in the inclusion sense. If two sets are homeomorphic, they have the same number of connected components. If $X \subseteq c(T)$, X is connected iff $|X|$ is connected in $\mathbb{R}^3$.

If X is a set of simplices with a same dimension k, X is *strongly connected* if for all σ and σ' in X, there are $\sigma_i \in X$ such that $1 \leq i \leq m$, $\sigma_1 = \sigma$, $\sigma_m = \sigma'$ and every $\sigma_i \cap \sigma_{i+1}$ is a $k-1$ dimensional simplex. For example, T^∞, T and ∂T are strongly connected. If X is strongly connected, X is connected.

Let $X \subset c(T)$ be a connected 2-manifold. The *genus* g of X meets $2(1-g) = v - e + t$ where v, e and t are the numbers of the vertices, edges and triangles in $c(X)$, respectively. Two connected 2-manifolds in $c(T)$ are homeomorphic iff they have the same genus. The genus is the number of handle(s) of X, e.g. $g = 0$ if X is a 2-sphere, $g = 1$ if X is homeomorphic to a torus. The *genus* of a non-connected 2-manifold in $c(T)$ is the sum of genuses of its connected components.

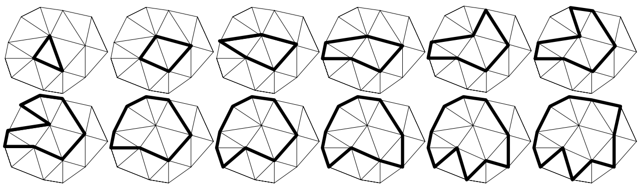

Fig. 1 A shelling example in the 2D case: O is a growing set of triangles such that ∂O is a cycle (drawn with bold edges).

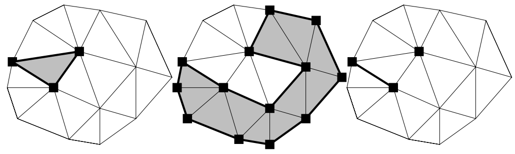

Fig. 2 Lemma 1 in the 2D case. Left: $c(\Delta)$ and $c(\partial\Delta)$. Middle: $c(O)$ and $c(\partial O)$. Right: $c(\Delta) \cap c(O)$. Both Δ and O are in gray, T is the set of all triangles, $\Delta \in T \setminus O$, the edges of $c(\partial\Delta)$ and $c(\partial O)$ are thickened, the vertices of $c(\Delta)$ and $c(O)$ are black squares. We see that $c(\Delta) \cap c(O)$ has 1 edge and 3 vertices, $c(\Delta) \cap c(O) = c(\partial\Delta) \cap c(\partial O)$, $\partial O \cap \partial\Delta$ has 1 edge.

3 Shelling definition and properties

First Sec. 3.1 provides our definition and explains the shelling algorithm in [13]. Then Sec. 3.2 describes the two cases of shelling steps. Last Sec. 3.3 details the topological changes in these two cases.

3.1 Definition

In our context (surface reconstruction), a *shelling* is a series of tetrahedron sets $O_0, O_1, \cdots, O_n$ such that

1. $O_0 \subseteq T^\infty$ and ∂O_0 is a non-empty 2-manifold
2. $O_{i+1} = O_i \cup \{\Delta_i\}$ where a tetrahedron $\Delta_i \in T \setminus O_i$ and $c(O_i) \cap c(\Delta_i)$ is a non-empty 2D-pure simplicial complex if $0 \leq i < n$.

Thus every ∂O_i is a 2-manifold thanks to Theorem 1. Fig. 1 shows a shelling in the 2D case to help intuition: replace tetrahedra by triangles, 2-manifold by 1-manifold (a disjoint union of cycles), 2D-pure by 1D-pure.

There is another definition in remark 8.3.ii of [25] (see also remark 2 in chapter 7 of [9], and definition 2.7 of [12]): a shelling of a d-D pure simplicial complex K is a series $\sigma_0, \sigma_1, \cdots, \sigma_n$ of d-D simplices such that $K = c(\sigma_0, \sigma_1, \cdots \sigma_n)$ and every $c(\sigma_0, \sigma_1, \cdots \sigma_{j-1}) \cap c(\sigma_j)$ is a non-empty $(d-1)$-D-pure simplicial complex. Both definitions are equivalent if $d = 3$ and $O_0 = \{\sigma_0\} \subseteq T$ and $K \subseteq c(T)$.

Our definition is different by two ways. First we prefer a cumulative formulation based on tetrahedron sets O_i since our application computes surfaces ∂O_i. Second a more general initialization O_0 is used since shellings are started at several steps of the surface reconstruction in [13]. Indeed,

Theorem 2 *Algorithm 1 in [13] generates a shelling.*

We describe this algorithm before the proof of Theorem 2. Let $F \subseteq T$ be the *free-space*: a tetrahedron $\Delta \in F$ iff there is a point $\mathbf{x}$ reconstructed from a view-point $\mathbf{c}$ using Computer Vision techniques such that $\Delta \cap \mathbf{cx} \neq \emptyset$. Let n_Δ be the number of line segments $\mathbf{cx}$ intersecting Δ. The goal is the computation of $O \subseteq F$ that maximizes $\sum_{\Delta \in O} n_\Delta$ subject to the constraint that ∂O is a 2-manifold. Let $Q \subseteq F \setminus O$ be a set of tetrahedra that should be added to the current estimate of O. Furthermore, Q is included in the immediate neighborhood of O: every tetrahedron $\Delta \in Q$ has a common triangle with a tetrahedron in O (if $O \neq \emptyset$), i.e. $\partial\Delta \cap \partial O \neq \emptyset$. At each step of Algorithm 1, we remove a tetrahedron Δ from Q with the largest n_Δ, then we add Δ to O if ∂O remains a 2-manifold. If Δ is added to O, we also add to Q every tetrahedron $\Delta' \in F \setminus O$ such that $\partial\Delta \cap \partial\Delta' \neq \emptyset$. This algorithm stops when $Q = \emptyset$. It is used several times in [13] for several initializations of Q and O.

Proof This algorithm computes a series O_i (consecutive values of O) such that $O_{i+1} = O_i \cup \{\Delta_i\}$, $\Delta_i \in T \setminus O_i$, $\partial\Delta_i \cap \partial O_i \neq \emptyset$ and ∂O_i is a 2-manifold for all i. According to Theorem 1, $c(O_i) \cap c(\Delta_i)$ is 2D pure. Since $\partial\Delta_i \cap \partial O_i \subseteq c(O_i) \cap c(\Delta_i)$, $c(O_i) \cap c(\Delta_i)$ is non-empty 2D pure. Furthermore ∂O_0 is a non-empty 2-manifold (if the algorithm starts with $O = \emptyset$, set O_0 as the first tetrahedron added to O). □

3.2 Basic properties

Several remarks can be done about shelling thanks to the following Lemma (this is Lemma 1 in [14]).

Lemma 1 *Let $O \subseteq T^\infty$ and Δ be a tetrahedron in $T \setminus O$. Then $c(O) \cap c(\Delta)$ is a simplicial complex in $\mathbb{R}^3$. Furthermore, $c(O) \cap c(\Delta) = c(\partial O) \cap c(\partial\Delta)$ and the triangles in $c(O) \cap c(\Delta)$ are exactly those in $\partial O \cap \partial\Delta$ (Fig. 2).*

First the shelling definition is the same if we replace $c(O_i) \cap c(\Delta_i)$ by $c(\partial O_i) \cap c(\partial\Delta_i)$. Intuitively, this is the simplicial complex "between" O_i and Δ_i. Second the number of the triangles in $\partial O_i \cap \partial\Delta_i$ cannot be 0.

Now we provide the two cases of shelling steps. They are in a corollary of the following theorem:

Theorem 3 *Let Δ be a tetrahedron in T. Let $K \subseteq c(\Delta)$ be a simplicial complex. Then K is a non-empty*

2D-pure simplicial complex iff it is a 2-ball or a 2-sphere. Let f be the number of the triangles in K. In the 2-ball case, $f \in \{1,2,3\}$. In the 2-sphere case, $f = 4$.

Proof We show that assertions A "K is a non-empty 2D-pure simplicial complex" and B "K is a 2-ball or a 2-sphere" are simultaneously true or false for every simplicial complex $K \subseteq c(\Delta)$.

If $K = \emptyset$ or $K = c(\Delta)$ or K has only one vertex, both A and B are wrong. Otherwise K has at least two vertices. If one of these vertices is not in an edge of K, K is not connected. Thus both A and B are wrong.

Now we consider the case where an edge $\mathbf{ab} \in K$ and triangles $\mathbf{abc} \notin K$ and $\mathbf{abd} \notin K$ using notation $\mathbf{abcd} = \Delta$. Here A is wrong. Let points $\mathbf{x} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ and $\mathbf{y} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$. Assume (reductio ad absurdum) that B is true. Thus $|K| \setminus \{\mathbf{x}, \mathbf{y}\}$ is homeomorphic to a 2-ball or a 2-sphere minus two points, which is connected. This is impossible since $|K| \setminus \{\mathbf{x}, \mathbf{y}\}$ has (at least) two connected components: $\mathbf{xy} \setminus \{\mathbf{x}, \mathbf{y}\}$ and another one included in $(\mathbf{ax} \setminus \{\mathbf{x}\}) \cup (\mathbf{by} \setminus \{\mathbf{y}\}) \cup \mathbf{cda} \cup \mathbf{cdb}$.

Assume that we are not in the previous cases. Thus A is true. We show that B is true. We have triangles $t_i \in \partial\Delta$ and $K = c(t_1, t_2, \cdots, t_f)$ where $f \geq 1$. If $f = 4$, $K = c(\partial\Delta)$ and K is a 2-sphere. If $f = 1$, $|K| = t_1$ is a 2-ball. If $f = 3$, $|K| = t_1 \cup t_2 \cup t_3$ is homeomorphic to t_4 (use the piecewise linear function that is the identity at the t_4 vertices and maps the t_4 center to vertex $t_1 \cap t_2 \cap t_3$). If $f = 2$, $|K| = t_1 \cup t_2$ is homeomorphic to t_3 (thanks to a similar piecewise linear function). □

Corollary 1 *A simplicial complex $c(O_i) \cap c(\Delta_i)$ is non-empty and 2D-pure (in the shelling definition) iff it is a 2-ball or a 2-sphere. Let f be the number of the triangles in $\partial O_i \cap \partial\Delta_i$. In the ball case, $1 \leq f \leq 3$, otherwise $f = 4$.*

Proof The set $K = c(O_i) \cap c(\Delta_i)$ is a simplicial complex (Lemma 1) included in $c(\Delta_i)$. According to Theorem 3, K is non-empty and 2D-pure iff it is a 2-ball ($f \in \{1,2,3\}$) or a 2-sphere ($f = 4$). □

3.3 Topology of surfaces ∂O_i

Theorem 4 describes all topology change(s) by adding one tetrahedron in O_i. The two following lemmas are needed in its proof (the former is Lemma 3 in [14].

Lemma 2 *Let $O \subset T^\infty$ and Δ be a tetrahedron in $T \setminus O$. Then $\partial(O \cup \{\Delta\}) = (\partial\Delta \setminus \partial O) \cup (\partial O \setminus \partial\Delta)$.*

Lemma 3 *Let $O \subset T^\infty$ be such that $\partial O \subset c(T)$ and ∂O is a 2-manifold. Let Δ be a tetrahedron in $T \setminus O$ such that $c(O) \cap c(\Delta)$ is 2D pure. Let f be the number of the triangles in $\partial O \cap \partial\Delta$. If $1 \leq f \leq 3$, ∂O and $\partial(O \cup \{\Delta\})$ are homeomorphic.*

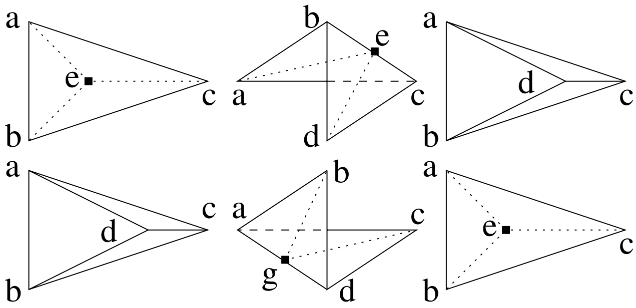

Fig. 3 Triangle splitting for Lemma 3 in cases $f = 1$ (left), $f = 2$ (middle) and $f = 3$ (right). There are $\partial O \cap \partial\Delta$ on the top and $\partial\Delta \setminus \partial O$ on the bottom (φ maps the former to the latter).

Proof We find a homeomorphism φ such that $\varphi(|\partial O|) = |\partial(O \cup \{\Delta\})|$. First φ is defined by its values on vertices and linear interpolation on the ∂O triangles (or their subdivisions) as follows. For every vertex $\mathbf{v}$ in $c(\partial O)$, we set $\varphi(\mathbf{v}) = \mathbf{v}$. We use notation $\Delta = \mathbf{abcd}$.

If $f = 1$, $\partial O \cap \partial\Delta = \{\mathbf{abc}\}$, we split $\mathbf{abc}$ into triangles $\mathbf{abe}$, $\mathbf{bce}$, $\mathbf{cae}$ where $\mathbf{e} = (\mathbf{a} + \mathbf{b} + \mathbf{c})/3$, and set $\varphi(\mathbf{e}) = \mathbf{d}$. We obtain $\varphi(\mathbf{abc}) = \mathbf{abd} \cup \mathbf{bcd} \cup \mathbf{cad}$.

If $f = 2$, $\partial O \cap \partial\Delta = \{\mathbf{abc}, \mathbf{bcd}\}$, we split $\mathbf{bc}$ into edges $\mathbf{be}$ and $\mathbf{ec}$ where $\mathbf{e} = (\mathbf{b} + \mathbf{c})/2$, we split $\mathbf{ad}$ into edges $\mathbf{ag}$ and $\mathbf{gd}$ where $\mathbf{g} = (\mathbf{a} + \mathbf{d})/2$, and set $\varphi(\mathbf{e}) = \mathbf{g}$. Note that this scheme also splits every triangle including $\mathbf{bc}$ or $\mathbf{ad}$ into two other triangles. We obtain $\varphi(\mathbf{abc} \cup \mathbf{bcd}) = \mathbf{adb} \cup \mathbf{adc}$.

If $f = 3$, $\partial O \cap \partial\Delta = \{\mathbf{dab}, \mathbf{dbc}, \mathbf{dca}\}$, we split $\mathbf{abc}$ into triangles $\mathbf{abe}$, $\mathbf{bce}$, $\mathbf{cae}$ where $\mathbf{e} = (\mathbf{a} + \mathbf{b} + \mathbf{c})/3$, and reset $\varphi(\mathbf{d}) = \mathbf{e}$. We obtain $\varphi(\mathbf{dab} \cup \mathbf{dbc} \cup \mathbf{dca}) = \mathbf{abc}$. The three cases $f \in \{1,2,3\}$ are in Fig. 3.

Second we check that every triangle $t \in \partial O \setminus \partial\Delta$ and $\varphi_{|t}$ (φ restricted to t) are unchanged by the modifications above. If $t \cap \Delta = \emptyset$, both t and $\varphi_{|t}$ are unchanged. Now we assume $t \cap \Delta \neq \emptyset$. Since $c(T)$ is a simplicial complex, $t \cap \Delta$ is a vertex or an edge of Δ. We consider several cases. If $f = 1$, $t \neq \mathbf{abc}$ and $\mathbf{e}$ is not a vertex of t. Thus both t and $\varphi_{|t}$ are unchanged. Assume (reductio ad absurdum) that $f = 2$ and $\mathbf{bc} \subset t$. Thus $\mathbf{bc}$ is a face of three distinct triangles t, $\mathbf{abc}$ and $\mathbf{bcd}$ in ∂O, although ∂O is a 2-manifold (impossible). Assume (reductio ad absurdum) that $f = 2$ and $\mathbf{ad} \subset t$. Thus $\mathbf{ad} \in c(O) \cap c(\Delta)$ although triangles $\mathbf{adb}$ and $\mathbf{adc}$ are not in $c(O) \cap c(\Delta)$, i.e. $c(O) \cap c(\Delta)$ is not 2D pure (impossible). Therefore both t and $\varphi_{|t}$ are unchanged if $f = 2$. Assume (reductio ad absurdum) that $f = 3$ and $\mathbf{d} \in t$. Thus $\{t\} \cup \mathbf{d} \times \textbf{a-b-c-a} \subseteq (\partial O)_{\mathbf{d}}$ and $t \notin \mathbf{d} \times \textbf{a-b-c-a}$, i.e. $\mathbf{d}$ is a singular vertex of ∂O (impossible). Therefore both t and $\varphi_{|t}$ are unchanged if $f = 3$.

Third we show that φ is surjective. Thanks to Lemma 2 and since $\varphi(|\partial O \cap \partial\Delta|) = |\partial\Delta \setminus \partial O|$ and φ is the identity

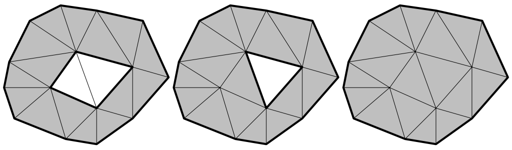

Fig. 4 Theorem 4 in the 2D case (O is gray). Left: O encloses a cavity defined by two white triangles. Middle: add a triangle to O such that $c(O) \cap c(\Delta)$ is a ball. Right: add a triangle to O such that $c(O) \cap c(\Delta)$ is a sphere. The number of the connected component(s) of ∂O is two (left and middle) or one (right).

in $|\partial O \setminus \partial\Delta|$,

$$\begin{aligned}\varphi(|\partial O|) &= \varphi(|\partial O \cap \partial\Delta| \cup |\partial O \setminus \partial\Delta|) &(1)\\ &= |\partial\Delta \setminus \partial O| \cup |\partial O \setminus \partial\Delta| &(2)\\ &= |\partial(O \cup \{\Delta\})|. &(3)\end{aligned}$$

Fourth we show that φ is a homeomorphism. We only need to check that φ maps distinct vertices to distinct vertices [19]. This is true if $f = 2$ and $f = 3$. Assume (reductio ad absurdum) that $f = 1$ and $\mathbf{d} \in c(\partial O)$. Thus $\mathbf{d} \in c(O) \cap c(\Delta)$. Since triangles **dab**, **dbc** and **dca** are not in $c(O) \cap c(\Delta)$, $c(O) \cap c(\Delta)$ is not 2D pure (impossible). If $f = 1$, $\mathbf{d} = \varphi(\mathbf{e})$ is different to all vertices in $c(\partial O)$ and we obtain the result. □

Note that Appendix B of [14] provides a first version of the proof of Lemma 3 without details on steps 2 and 4.

Theorem 4 *In a shelling, all ∂O_i are 2-manifolds with the same genus. If $c(O_i) \cap c(\Delta_i)$ is a 2-ball, ∂O_{i+1} and ∂O_i have the same number of connected components. If $c(O_i) \cap c(\Delta_i)$ is a 2-sphere, ∂O_i has one connected component more than ∂O_{i+1} (Fig. 4).*

Proof Every ∂O_i is a 2-manifold thanks to Theorem 1 and since ∂O_0 is a 2-manifold. Let f be the number of the triangles in $\partial O_i \cap \partial\Delta_i$. If $f \in \{1, 2, 3\}$, ∂O_{i+1} and ∂O_i are homeomorphic thanks to Lemma 3. Therefore ∂O_{i+1} and ∂O_i have the same genus and the same number of connected component(s).

Now we study case $f = 4$. We have $\partial\Delta_i \subseteq \partial O_i$ and Lemma 2 implies $\partial O_{i+1} = \partial O_i \setminus \partial\Delta_i$. Therefore $\partial O_i = \partial O_{i+1} \cup \partial\Delta_i$ and $\partial O_{i+1} \cap \partial\Delta_i = \emptyset$. Assume (reductio ad absurdum) that $\partial\Delta_i$ is not a connected component of ∂O_i. Thus these is an edge $\mathbf{v}\mathbf{v}' \in c(\partial O_i)$ such that $\mathbf{v} \in c(\partial\Delta_i)$ and $\mathbf{v}' \notin c(\partial\Delta_i)$. This implies that there is a triangle in $\partial O_i \setminus \partial\Delta_i$ that includes $\mathbf{v}\mathbf{v}'$. Therefore $\mathbf{v} \in c(\partial\Delta_i) \cap c(\partial O_{i+1})$. Since ∂O_{i+1} is a 2-manifold, there is a cycle $\mathbf{a}_*$ such that $(\partial O_{i+1})_\mathbf{v} = \mathbf{v} \times \mathbf{a}_*$. Thus $(\mathbf{v} \times \mathbf{a}_*) \cup (\partial\Delta_i)_\mathbf{v} \subseteq (\partial O_i)_\mathbf{v}$ and $(\partial\Delta_i)_\mathbf{v} \setminus \mathbf{v} \times \mathbf{a}_* \neq \emptyset$, which imply that $\mathbf{v}$ is a singular vertex of ∂O_i (impossible). Since $\partial\Delta_i$ is a connected component of ∂O_i and $\partial O_i = \partial O_{i+1} \cup \partial\Delta_i$, ∂O_i has one connected component more than ∂O_{i+1}. Last ∂O_{i+1} and ∂O_i have the same genus since $\partial\Delta_i$ has zero genus. □

In most cases, $c(O_i) \cap c(\Delta_i)$ is a 2-ball. In the few cases where Δ_i is a cavity of O_i (O_0 can be like a piece of Swiss cheese that has cavities as in Fig. 4), $c(O_i) \cap c(\Delta_i)$ is a 2-sphere.

Corollary 2 describes a frequent case where shelling does not change the topology of ∂O_i. We need

Lemma 4 *If $\emptyset \neq O \subsetneq T^\infty$, $\partial O \neq \emptyset$.*

The proof of Lemma 4 is straightforward (T^∞ is strongly connected) and is in the supplementary material.

Corollary 2 *Assume that $O_i \subset T^\infty$ and ∂O_i is a 2-sphere in a shelling. If $i \leq j < n$ and $O_{j+1} \neq T^\infty$, $c(O_j) \cap c(\Delta_j)$ is a 2-ball and ∂O_{j+1} is a 2-sphere.*

Proof We assume (induction) that ∂O_j is a 2-sphere (this is OK if $j = i$) and show that $c(O_j) \cap c(\Delta_j)$ is a 2-ball and ∂O_{j+1} is a 2-sphere if $O_{j+1} \neq T^\infty$.

Assume (reductio ad absurdum) that $c(O_j) \cap c(\Delta_j)$ is a 2-sphere. Since ∂O_j has one connected component, Theorem 4 implies that ∂O_{j+1} does not have a connected component, i.e. $\partial O_{j+1} = \emptyset$. Since $\emptyset \neq O_{j+1} \subsetneq T^\infty$, this contradicts Lemma 4.

Thus $c(O_j) \cap c(\Delta_j)$ is a 2-ball (Corollary 1) and ∂O_{j+1} is a 2-manifold with the same genus and the same number of connected component than ∂O_j (Theorem 4), i.e. ∂O_{j+1} is a 2-sphere. □

4 Overview of 3-ball shellability

This section summarizes previous works in Combinatorial Topology. Here we would like to know if $O \subseteq T$ is shellable, i.e. if O can be the end value of a shelling started from a single tetrahedron. First Sec. 4.1 describes and compare several kinds of shellability and non-shellability. Second Sec. 4.2 presents invariant numbers and a necessary condition for the O shellability. Last Sec. 4.3 and 4.4 list positive and negative cases of shellability.

4.1 Definitions

According to Theorem 5 (supplementary material),

Theorem 5 *Let $O \subseteq T$. Then O is a 3-ball iff ∂O is a 2-sphere.*

assertions "O is a 3-ball" and "∂O is a 2-sphere" are equivalent in our context. Although the introduction of 3-balls seems useless, we mention it (and use it) since this formulation is often used in the bibliography.

A tetrahedron set $O \subseteq T$ is *shellable* if there is a tetrahedron $\Delta \in T$ and a shelling $O_0 \cdots O_n$ such that $O_0 = \{\Delta\}$ and $O = O_n$. We also say that $O_0 \cdots O_n$ is a *shelling of O*. Thanks to Corollary 2, we note that all ∂O_i are 2-spheres and all $c(O_i) \cap c(\Delta_i)$ are 2-balls. Thus O is a 3-ball.

A 3-ball $O \subseteq T$ is *extendably shellable* if for every shellable $O' \subset O$, there is a shelling $O_0 \cdots O_k$ such that $O_0 = O'$ and $O_k = O$. In other words, every partial shelling of O can be completed to reach O. Therefore O is shellable if O is extendably shellable.

Extendable shellability is a convenient case of shellability. If O is extendably shellable and if a greedy algorithm constructs a shelling of O by choosing successive tetrahedra in O (e.g. Algorithm 1 in [13] using $F = O$), the algorithm always succeeds, i.e. its final value is always $O_n = O$. If O is shellable but non-extendably shellable, the greedy algorithm can choose tetrahedra at the shelling beginning such that it "gets stuck". This means that there is $O_i \subsetneq O$ in the shelling computation such that $c(O_i) \cap c(\Delta)$ is never non-empty 2D-pure for every tetrahedron Δ tried by the algorithm (thus the algorithm stops). Thanks to Theorem 1, this means that $\partial(O_i \cup \{\Delta\})$ is not a 2-manifold (i.e. $\partial(O_i \cup \{\Delta\})$ has a singular vertex) or $c(O_i) \cap c(\Delta) = \emptyset$.

A 3-ball $O \subseteq T$ is *strongly non-shellable* if O has at least two tetrahedra and if for every tetrahedron $\Delta \in O$, $c(\Delta) \cap c(\partial O)$ is not a 2-ball. Here we note that $\Delta \in O$ in contrast to the shelling definition where $\Delta_i \notin O_i$. According to Proposition 2.4.iv in [24], O is non-shellable if O is strongly non-shellable (principle of the proof: if $O_0 \cdots O_n$ is a shelling of O, we have $O = O_{n-1} \cup \{\Delta\}$, and see that $c(\partial O) \cap c(\Delta)$ is a 2-ball as the complement of 2-ball $c(\partial O_{n-1}) \cap c(\Delta)$ in 2-sphere $c(\partial \Delta)$).

Strongly non-shellability is an easy case of non-shellability for several reasons. First it is not difficult to check that a given O is strongly non-shellable. In comparison, it is quite more difficult to check that O is non-shellable: we don't know whether a polynomial-time algorithm can do that [8]. Second, a strongly non-shellable O can be build with a few tetrahedra. Third, non-shellability and strongly non-shellability are related by Proposition 2.4.v in [24]: if O is non-shellable, there is a strongly non-shellable 3-ball $O' \subseteq O$ (principle of the proof: construct a shelling of O in the reverse order if every 3-ball $O' \subseteq O$ is not strongly non-shellable). More details about the relations between strongly non-shellability and non-shellability are in the supplementary material.

4.2 h-numbers of O

The result described in this section is known for every dimension (Theorem 8.19 in [25], see also Proposition 7.7 in [9]). Here we present a simpler (specialized) proof in our 3D case.

Let O be a 3-ball in T and f_0 (respectively, f_1, f_2 and f_3) be the number of the vertices (respectively, edges, triangles and tetrahedra) in $c(O)$. According to [25, 9], the definition of h-numbers h_1, h_2 and h_3 of O is

$$\begin{pmatrix}1\\h_1\\h_2\\h_3\end{pmatrix} = \mathtt{H}\begin{pmatrix}1\\f_0\\f_1\\f_2\end{pmatrix} \text{ where } \mathtt{H} = \begin{pmatrix}+1&0&0&0\\-4&+1&0&0\\+6&-3&+1&0\\-4&+3&-2&+1\end{pmatrix}. \quad (4)$$

If O is shellable, we classify the shelling steps in three cases and all steps in a case update the numbers of the simplices by the same way. Thanks to this, Theorem 6 shows that the number of the steps in the j-th case is a function of the f_is, more precisely it is h_j. The number of the steps in the j-th case is invariant to the shelling choice since h_j does not depend on it.

Theorem 6 *If O is shellable, every shelling of O has h_j step(s) $O_{i+1} = O_i \cup \{\Delta_i\}$ such that the number of the triangles in $\partial O_i \cap \partial \Delta_i$ is equal to j.*

Proof If $X \subseteq c(T)$, we define the vector $f(X) \in \mathbb{N}^4$ whose k-th coordinate is the number of $k-1$-dimensional simplices in $c(X)$. Let $O_0 \cdots O_n$ be a shelling of O and the tetrahedron $\Delta_i \in T \setminus O_i$ such that $O_{i+1} = O_i \cup \{\Delta_i\}$. We will calculate $\delta_i = f(c(O_{i+1})) - f(c(O_i))$.

Since $c(O_{i+1}) = c(O_i) \cup (c(\Delta_i) \setminus c(O_i))$, $\delta_i = f(c(\Delta_i) \setminus c(O_i))$. Let n_i be the number of the triangles in $\partial O_i \cap \partial \Delta_i$ and $\mathbf{abcd} = \Delta_i$. Since $c(O_i) \cap c(\Delta_i)$ is a 2-ball, there are three cases according to Corollary 1.

If $n_i = 1$, $c(O_i) \cap c(\Delta_i) = c(\mathbf{abc})$. Thus $c(\Delta_i) \setminus c(O_i) = \{\mathbf{d}, \mathbf{da}, \mathbf{db}, \mathbf{dc}, \mathbf{dab}, \mathbf{dbc}, \mathbf{dca}, \mathbf{abcd}\}$. We obtain $\delta_i = \begin{pmatrix}1 & 3 & 3 & 1\end{pmatrix}^T$.

If $n_i = 2$, $c(O_i) \cap c(\Delta_i) = c(\mathbf{abc}, \mathbf{bcd})$. Thus $c(\Delta_i) \setminus c(O_i) = \{\mathbf{ad}, \mathbf{adb}, \mathbf{adc}, \mathbf{abcd}\}$. We obtain $\delta_i = \begin{pmatrix}0 & 1 & 2 & 1\end{pmatrix}^T$.

If $n_i = 3$, $c(O_i) \cap c(\Delta_i) = c(\mathbf{abc}, \mathbf{acd}, \mathbf{adb})$. Thus $c(\Delta_i) \setminus c(O_i) = \{\mathbf{bcd}, \mathbf{abcd}\}$. We obtain $\delta_i = \begin{pmatrix}0 & 0 & 1 & 1\end{pmatrix}^T$.

Let $j \in \{1, 2, 3\}$ and h'_j be the number of the step(s) $O_{i+1} = O_i \cup \{\Delta_i\}$ such that $n_i = j$. Let

$$\mathtt{A} = \begin{pmatrix}4&1&0&0\\6&3&1&0\\4&3&2&1\\1&1&1&1\end{pmatrix} \text{ and } \mathtt{B} = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&1&-1&1\end{pmatrix}. \quad (5)$$

Since O_0 has one tetrahedron, $f(c(O_0)) = \begin{pmatrix}4 & 6 & 4 & 1\end{pmatrix}^T$. Now we have

$$f(c(O)) = f(c(O_0)) + \sum_{i=0}^{n-1} \delta_i = \mathtt{A}\begin{pmatrix}1 & h'_1 & h'_2 & h'_3\end{pmatrix}^\top. \quad (6)$$

Furthermore, Eqs. 6 and 5 imply

$$f_0 - f_1 + f_2 - f_3 = \begin{pmatrix}1 & -1 & 1 & -1\end{pmatrix} f(c(O)) = 1, \quad (7)$$

which in turn implies

$$f(c(O)) = \begin{pmatrix}f_0 & f_1 & f_2 & f_3\end{pmatrix}^\top = \mathtt{B}\begin{pmatrix}1 & f_0 & f_1 & f_2\end{pmatrix}^\top. \quad (8)$$

Thanks to Eqs. 6 and 8 and since $\mathtt{H} = \mathtt{A}^{-1}\mathtt{B}$, we obtain

$$\begin{pmatrix}1 & h'_1 & h'_2 & h'_3\end{pmatrix}^\top = \mathtt{H}\begin{pmatrix}1 & f_0 & f_1 & f_2\end{pmatrix}^\top. \quad (9)$$

Therefore $h'_j = h_j$ for $j \in \{1, 2, 3\}$. □

We also see that a necessary condition for shellability is $h_1 \geq 0$ and $h_2 \geq 0$ and $h_3 \geq 0$ (O is non-shellable if there is j such that $h_j < 0$). Furthermore, the h_j are consistent even if O is non-shellable:

Property 1 If $O \subseteq T$ is a 3-ball, $1 + h_1 + h_2 + h_3 = f_3$.

Proof According to Eq. 4,

$$1 + h_1 + h_2 + h_3 = \begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix}\mathtt{H}\begin{pmatrix}1 & f_0 & f_1 & f_2\end{pmatrix}^\top \quad (10)$$

$$= -1 + f_0 - f_1 + f_2. \quad (11)$$

Last we obtain the result thanks to Euler's relation for 3-balls [9], i.e. $f_0 - f_1 + f_2 - f_3 = 1$. □

4.3 Positive cases

First, all 3-balls with less than 9 vertices are extendably shellable and almost all 3-balls with 9 vertices are shellable [17] (there are only 29 non-shellable 3-balls over the 2451305 combinatorial 3-balls with 9 vertices). Second, every 3D Delaunay T is a shellable 3-ball [24,3]. Thus every $O \subseteq T$ such that $|O|$ is convex is shellable. Indeed, such an O is the set of all tetrahedra of the 3D Delaunay triangulation defined by using only the vertices in $c(O)$. Third Sec. 5 shows that $O \subseteq T$ is shellable if $|O|$ is star-shaped. This generalizes the convex case above, since a convex set is star-shaped with respect to every point in the convex set.

4.4 Negative cases

There is a survey [24] of non-shellable 3-balls in a more general context than ours: some of them are constructed using a polytopal complex instead of a simplicial complex, i.e. the tetrahedra are replaced by convex hulls with $n \geq 4$ vertices likes cubes. Since we focus on 3-balls included in a 3D Delaunay triangulation, they should be converted to analog examples in a 3D Delaunay. There are two kinds of examples: strongly non-shellable 3-balls defined by a small (ideally minimal) set of simplices, and non-shellable 3-balls defined by an important (not explicit) set of simplices and a global property. In the former, there is a 3-ball that has only 10 vertices and 21 tetrahedra (or 12 vertices and 25 tetrahedra). In the latter, there is a 3-ball called "Knotted hole ball" (K); K is non-shellable thanks to a property based on the knot theory: there is a knotted curve (a cycle of edges that cannot be included in a 2-sphere) with all edges on the boundary of K except one edge that is in K.

There are other negative cases. A 3D Delaunay triangulation T can be non-extendably shellable [24]. Thus a greedy shelling algorithm can fail and miss O if $O \subseteq T$ and $|O|$ is convex. Last Sec. 6 provides a family of strongly non-shellable 3-balls that are not difficult to visualize in 3D. It includes the 3-ball with 12 vertices and 25 tetrahedra in [24], which is in a 3D Delaunay triangulation.

5 Star-shape shelling

In Sec. 5, there are tetrahedron sets O, O' and a point $\mathbf{c} \in |O|$ such that $O \subsetneq O' \subseteq T$ and both O' and O are $\mathbf{c}$-star-shaped. We show that there is a shelling $O_0 \cdots O_n$ such that $O_0 = O$ and $O_n = O'$. Such a result is suggested in [7] if $O' = T$ and O has a single tetrahedron. Here we need a non-degenerate case: $\mathbf{c}$ is not in a plane including a triangle of $c(T)$.

5.1 Summary

The proof has three main steps. First Sec. 5.2 provides useful lemmas and a star-shape criterion for a set of tetrahedra included in T (Theorem 7). Second Sec. 5.3 describes a visibility relation between two tetrahedra in T with respect to a view point $\mathbf{c}$: Δ is behind Δ'. Then Sec. 5.3 shows that there is a "front" tetrahedron in $O' \setminus O$ for this relation (Theorem 8), i.e. this tetrahedron is not behind another one in $O' \setminus O$. Third Sec. 5.4 shows that the front tetrahedron is added to O such that we obtain a shelling step (Theorem 9) thanks to Theorems 7 and 8 and lemmas. Then Sec. 5.4 shows the existence of a shelling between O and O' (Corollary 3).

Last Sec. 5.5 presents a qualitative argument based on Corollary 3 to explain why the first shelling in [13] does not have excessive blocking.

5.2 Prerequisites

We give definitions before lemmas and theorems. If $\mathbf{a}$ and $\mathbf{b}$ are distinct points in $\mathbb{R}^3$, we distinguish the line

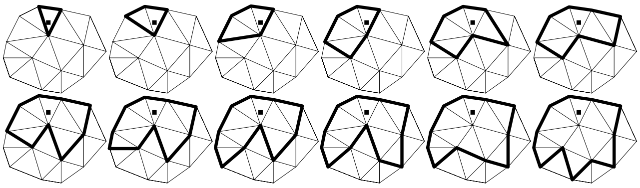

Fig. 5 A star-shape shelling example in the 2D case: O is a growing set of triangles such that ∂O is a cycle (bold edges) and O is always star-shaped with respect to the black dot.

segment **ab** and the full line (**ab**) including **a** and **b**:

$$\mathbf{ab} = \{(1-\lambda)\mathbf{a} + \lambda\mathbf{b}, 0 \leq \lambda \leq 1\} \text{ and} \quad (12)$$

$$(\mathbf{ab}) = \{(1-\lambda)\mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}. \quad (13)$$

A point $\mathbf{d}$ *is between* $\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{d} \in \mathbf{ab} \setminus \{\mathbf{a}, \mathbf{b}\}$.

A set $X \subset \mathbb{R}^3$ is *star-shaped* with respect to $\mathbf{c}$ (or X is $\mathbf{c}$-*star-shaped*) if $\mathbf{x} \in X$ implies that $\mathbf{xc} \subseteq X$. We say that $O \subseteq T$ is $\mathbf{c}$-*star-shaped* if $|O|$ is $\mathbf{c}$-*star-shaped*. If t is a triangle in $\mathbb{R}^3$, the *half-spaces* of t are the two half-spaces of $\mathbb{R}^3$ separated by the plane including t (each half-space includes the plane itself). Note that our non degenerate case implies that $\mathbf{c}$ is in exactly one half-space of every triangle in $c(T)$. Fig. 5 shows a star-shape shelling in the 2D case (replace tetrahedron by triangle etc, as in Fig. 1).

We also use other notations in Sec. 5. Let σ be a simplex in $\mathbb{R}^3$ and $\mathring{\sigma} = \sigma \setminus |\partial\sigma|$, i.e. $\mathring{\sigma}$ is the *interior* of σ. Let $B(\mathbf{x}, \epsilon)$ be the 3D ball centered at $\mathbf{x} \in \mathbb{R}^3$ with the radius $\epsilon > 0$. If σ is a tetrahedron in $\mathbb{R}^3$, $\mathring{\sigma}$ is an open set in $\mathbb{R}^3$, i.e. every point $\mathbf{x} \in \mathring{\sigma}$ is such that there is $\epsilon > 0$ and $B(\mathbf{x}, \epsilon) \subseteq \mathring{\sigma}$.

We need the three following lemmas to show Theorem 7 (the star-shape criterion) and others. The first two help to escape from degenerate configurations. The third one is a ray-tracing lemma, which provides a series of adjacent tetrahedra in T covering a line segment between two points in $|T|$.

Lemma 5 *If a tetrahedron $\Delta \in T$ and a point $\mathbf{x} \in \mathring{\Delta}$, Δ is the only tetrahedron in T that contains $\mathbf{x}$, and $\mathbf{x}$ is not in a triangle of $c(T)$. If a triangle $t \in c(T)$ and $\mathbf{x} \in \mathring{t}$, t is the only triangle in $c(T)$ that contains $\mathbf{x}$.*

The proof of Lemma 5 is straightforward ($c(T)$ is a simplicial complex) and is in the supplementary material.

Lemma 6 *Let $A \subseteq T$ and points $\mathbf{c} \in |A|$, $\mathbf{x} \in |A| \setminus \mathbf{c}$, $\mathbf{y} \in \mathbf{cx}$ such that $\mathbf{y} \notin |A|$. Let Δ_x be a tetrahedron in A such that $\mathbf{x} \in \Delta_x$. There is a point $\tilde{\mathbf{x}} \in \mathring{\Delta}_x$ and a tetrahedron $\Delta_y \in T \setminus A$ and a point $\tilde{\mathbf{y}} \in \mathring{\Delta}_y \cap \mathbf{c}\tilde{\mathbf{x}}$ such that $(\mathbf{c}\tilde{\mathbf{x}})$ does not intersect the edges in $c(T)$ (Fig. 6).*

Lemma 7 *Let Δ_x be a tetrahedron in T and points $\mathbf{x} \in \mathring{\Delta}_x$, $\mathbf{y} \in T \setminus \{\mathbf{x}\}$ such that $(\mathbf{xy})$ does not intersect the edges in $c(T)$. Let $\mathbf{x}_\lambda = (1-\lambda)\mathbf{x} + \lambda\mathbf{y}$ if $\lambda \in \mathbb{R}$. There are integer $k \geq 1$, reals $\lambda_1 \cdots \lambda_{k+1}$, tetrahedra $\Delta_1 \cdots \Delta_k$ in T such that $\lambda_i < \lambda_{i+1}$ if $1 \leq i \leq k$, $\mathbf{x}_\lambda \in \Delta_i$ iff $\lambda \in [\lambda_i, \lambda_{i+1}]$, every $\Delta_i \cap \Delta_{i+1}$ is a triangle. If $\Delta \in T$ and $(\mathbf{xy}) \cap \Delta \neq \emptyset$, Δ is a Δ_i and $(\mathbf{xy}) \cap \mathring{\Delta} \neq \emptyset$ (Fig. 7).*

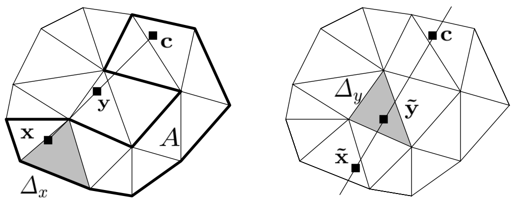

Fig. 6 Lemma 6 in the 2D case. Left: A is enclosed by the bold edges, $\mathbf{c} \in |A|$, $\mathbf{x} \in \Delta_x \in A$, Δ_x is grey, $(\mathbf{cx})$ intersects a vertex of $c(T)$ (an edge in 3D), $\mathbf{y} \in \mathbf{cx} \setminus |A|$. Right: replace $\mathbf{x}$ by $\tilde{\mathbf{x}} \in \mathring{\Delta}_x$ such that $(\mathbf{c}\tilde{\mathbf{x}})$ does not intersect the $c(T)$ vertices, replace $\mathbf{y}$ by $\tilde{\mathbf{y}} \in \mathring{\Delta}_y \cap \mathbf{c}\tilde{\mathbf{x}}$ such that $\Delta_y \in T \setminus A$, Δ_y is grey.

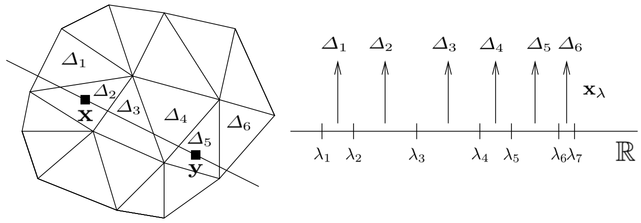

Fig. 7 Lemma 7 in the 2D case. Left: $(\mathbf{xy})$ successively intersects $\Delta_1 \cdots \Delta_6$ in T, every $\Delta_i \cap \Delta_{i+1}$ is an edge (a triangle in 3D). Right: $\mathbf{x}_\lambda$ maps $[\lambda_i, \lambda_{i+1}]$ in Δ_i.

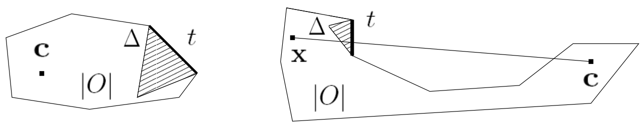

Fig. 8 Theorem 7 in the 2D case (tetrahedra are drawn as triangles). The streaked triangle is Δ and the bold edge is t. If O is $\mathbf{c}$-star-shaped (left), Δ and $\mathbf{c}$ are in the same t half-space. If O is not $\mathbf{c}$-star-shaped (right), there is $\mathbf{x} \in |O|$ such that we have not $\mathbf{xc} \subset |O|$. Thus there are Δ and t such that Δ and $\mathbf{c}$ are not in the same t half-space.

The proofs ot Lemma 6 and 7 are in Appendices A.1 and A.2). Theorem 7 converts the star-shape definition to a more tractable condition in our proof.

Theorem 7 *Let $O \subseteq T$ and a point $\mathbf{c} \in |O|$. Then O is $\mathbf{c}$-star-shaped iff every triangle $t \in \partial O$ is a face of a tetrahedron $\Delta \in O$ such that Δ and $\mathbf{c}$ are in the same t half-space (Fig. 8).*

This is similar to star-shape tests in [20,22], which use outward-pointing normals at points on a volume boundary and a scalar product instead of our condition based on simplices. The proof is summarized in Fig. 8 and detailed in Appendix A.3.

5.3 Visibility in a 3D Delaunay triangulation

Let tetrahedra $\Delta \in T$ and $\Delta' \in T$. Let l be a half-line starting at $\mathbf{c}$. We say that Δ *is behind* [7] Δ' if there is l such that $l \cap \mathring{\Delta} \neq \emptyset$ and $l \cap \mathring{\Delta}' \neq \emptyset$ and $\Delta \neq \Delta'$ and every point in $(l \setminus \{\mathbf{c}\}) \cap \mathring{\Delta}'$ is between $\mathbf{c}$ and every point of $(l \setminus \{\mathbf{c}\}) \cap \mathring{\Delta}$. Fig. 9 shows $\mathbf{c}$, l, Δ and Δ'.

The goal of Sec. 5.3 is Theorem 8: the existence of a "front" tetrahedron in $O' \setminus O$, i.e. a tetrahedron which is not behind another one in $O' \setminus O$. The proof is based on the fact that the relation "is behind" is acyclic in the 3D Delaunay triangulation T [7]. It also needs the two following lemmas to generate "behind" relations.

Lemma 8 *Assume that O' is $\mathbf{c}$-star-shaped and $\Delta \in O' \setminus O$ has a triangle face $t \notin \partial O$ such that Δ and $\mathbf{c}$ are not in the same t half-space. There is $\Delta' \in O' \setminus O$ such that $\partial\Delta' \cap \partial O \neq \emptyset$ and Δ is behind Δ' (Fig. 9).*

A sketch proof is provided before the proof for intuition. Let $\mathbf{y}$ be a point in Δ. Since O' is $\mathbf{c}$-star-shaped, $\mathbf{y} + \lambda(\mathbf{c}-\mathbf{y})$ successively intersects tetrahedra $\Delta_0, \Delta_1 \cdots \Delta_k$ in O' when λ increases from 0 to 1, and every $\Delta_i \cap \Delta_{i+1}$ is not empty. We choose $\mathbf{y}$ such that $\Delta_i \cap \Delta_{i+1}$ is a triangle for every i ($\mathbf{cy}$ does not intersect the edges in $c(T)$). Since $\mathbf{y} \in \Delta = \Delta_0 \in O' \setminus O$ and $\mathbf{c} \in \Delta_k \in O$, there is i such that $\Delta_i \in O' \setminus O$ and $\Delta_{i+1} \in O$. We obtain $\Delta' = \Delta_i$ (we have $\Delta \neq \Delta'$ if $\partial\Delta \cap \partial O = \emptyset$).

Proof First we set $A = O \cup \{\Delta\}$ and find points $\mathbf{x}'$ and $\mathbf{y}'$ that meet the assumptions of Lemma 6. Let $\mathbf{x}'$ be the barycentre of t and $\Delta_x = \Delta \in A$. If $t \in \partial T$ (reductio ad absurdum), both Δ and $\mathbf{c}$ are in the same t half-space since T is convex (impossible). Thus there is a tetrahedron $\Delta'' \in T \setminus \{\Delta\}$ such that $t = \Delta \cap \Delta''$. Since $t \notin \partial O$ and $\Delta \notin O$, $\Delta'' \in T \setminus O$ and thus $\Delta'' \in T \setminus A$. Since $\mathbf{c}$ and Δ are in different half-spaces of t, there is a point $\mathbf{y}' \in \mathbf{cx}' \cap \mathring{\Delta}''$. Since $\mathbf{y}'$ cannot be in a tetrahedron of A (Lemma 5), $\mathbf{y}' \notin |A|$.

Second we find a tetrahedron series covering $\mathbf{xc}$ ($\mathbf{x} \in \Delta$) using Lemma 7. According to Lemma 6, there are a tetrahedron $\Delta_y \in T \setminus A$ and points $\mathbf{x} \in \mathring{\Delta}_x$ and $\mathbf{y} \in \mathring{\Delta}_y \cap \mathbf{cx}$ such that $(\mathbf{cx})$ does not intersect the edges in $c(T)$. Now we have a tetrahedron series $\Delta_i \in T$ that covers $|T| \cap (\mathbf{xy})$ as defined by Lemma 7. Since $\mathbf{xc} \subseteq |T|$ (indeed $|T|$ is convex and include both $\mathbf{x}$ and $\mathbf{c}$) and $(\mathbf{xc}) = (\mathbf{xy})$, the tetrahedron series also covers $\mathbf{xc}$.

Third we find distinct Δ_i that include $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{c}$. There are j, n and l such that $\Delta_j = \Delta_x$, $\Delta_n = \Delta_y$ and $\mathbf{c} \in \Delta_l \in O$ (since $\mathbf{c} \in |O|$). Since $\mathbf{y} \in \mathbf{xc}$ and $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{y} = \mathbf{x}_1$, we have $j \leq n \leq l$. Since $\Delta_j \in A$ and $\Delta_n \notin A$ and $\Delta_l \in A$, we have $j < n < l$.

Fourth we show that $j \leq m \leq l$ implies $\Delta_m \in O'$. Since O' is $\mathbf{c}$-star-shaped and both $\mathbf{x}$ and $\mathbf{c}$ are in $|O'|$, $\mathbf{cx} \in |O'|$. Thus $\mathring{\Delta}_m \cap |O'| \neq \emptyset$ (Lemma 7) and $\Delta_m \in O'$ (Lemma 5).

Last we conclude. Since $\Delta_n \in O' \setminus O$ and $\Delta_l \in O$, there is m such that $\Delta_m \in O' \setminus O$ and $\Delta_{m+1} \in O$ and $n \leq m < l$. Let $\Delta' = \Delta_m$. The triangle $\Delta_m \cap \Delta_{m+1}$ is in $\partial\Delta' \cap \partial O$. Note that $\mathbf{x}_\lambda \in \Delta$ iff $\lambda \in [\lambda_j, \lambda_{j+1}]$, $\mathbf{x}_\lambda \in \Delta'$ iff $\lambda \in [\lambda_m, \lambda_{m+1}]$, $\lambda_{j+1} \leq \lambda_m$ (since $j < n \leq m$), $\mathbf{c} = \mathbf{x}_\mu$ with $\mu \in [\lambda_l, \lambda_{l+1}]$ and $\lambda_{m+1} \leq \lambda_l$ (since $m < l$). Thus Δ is behind Δ'. □

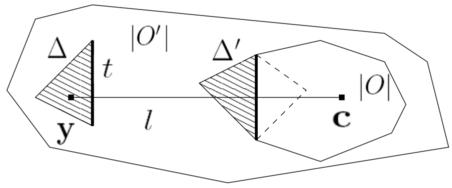

Fig. 9 Notations for Lemma 8 in the 2D case (tetrahedra are drawn as triangles). The two streaked triangles are Δ and Δ', the triangle with dotted edges is in O. Here Δ is behind Δ'.

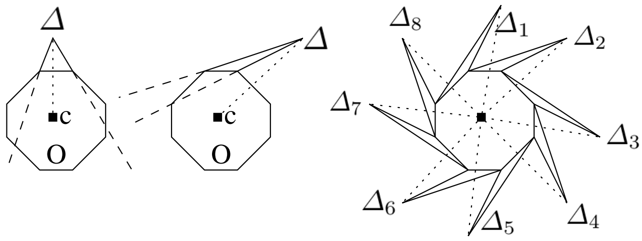

Fig. 10 Theorem 8 in the 2D case where O forms an octagon that is $\mathbf{c}$-star-shaped. Left: Δ meets all conditions of the theorem. Middle: Δ does not. Right: Δ never does (reductio ad absurdum). We see thatΔ_1 is behind Δ_2, which is behind Δ_3, $\cdots$ which is behind Δ_8, which is behind Δ_1. However such a cycle is impossible in a Delaunay triangulation [7].

Lemma 9 *If O' is $\mathbf{c}$-star-shaped and $O' \setminus O \neq \emptyset$, there is a tetrahedron $\Delta' \in O' \setminus O$ such that $\partial\Delta' \cap \partial O \neq \emptyset$.*

Proof Let Δ be a tetrahedron in $O' \setminus O$. If $\partial\Delta \cap \partial O \neq \emptyset$, we obtain $\Delta' = \Delta$. Otherwise, $\partial\Delta \cap \partial O = \emptyset$. Note that $\mathbf{c} \notin \Delta$ ($\mathbf{c} \notin \partial\Delta$ in our non degenerate case and $\mathbf{c} \notin \mathring{\Delta}$ by Lemma 5 and since $\mathbf{c} \in |O|$). Thus there is a triangle $t \in \partial\Delta$ such that $\mathbf{c}$ and Δ are not in the same t half-space. Since $t \notin \partial O$, Lemma 8 provides Δ'. □

Theorem 8 *There is a tetrahedron $\Delta \in O' \setminus O$ such that $\partial\Delta \cap \partial O \neq \emptyset$ and every triangle $t \in \partial\Delta \setminus \partial O$ is such that $\mathbf{c}$ and Δ are in the same t half-space (Fig. 10).*

Proof Assume (reductio ad absurdum) that

$$\forall \Delta \in O' \setminus O, \partial\Delta \cap \partial O = \emptyset \text{ or } \exists t \in \partial\Delta \setminus \partial O \text{ such that } \Delta \text{ and } \mathbf{c} \text{ are not in the same half-space of } t.$$

Thus $\Delta \in O' \setminus O$ and $\partial\Delta \cap \partial O \neq \emptyset$ imply that Δ meets the assumptions of Lemma 8.

The principle of the proof is the following. We find an infinite series of tetrahedra $\Delta_i \in O' \setminus O$ such that $\partial\Delta_i \cap \partial O \neq \emptyset$ and Δ_i is behind Δ_{i+1} for all i using Lemma 8. Since O' is finite, there are $m \neq n$ such that $\Delta_m = \Delta_n$, i.e. the relation "is behind" has a cycle. This contradicts [7] which shows that this relation is acyclic in Delaunay triangulation T.

Here is the series. First Lemma 9 provides $\Delta_0 \in O' \setminus O$ such that $\partial\Delta_0 \cap \partial O \neq \emptyset$. Now Δ_{i+1} is defined from Δ_i as follow. If $\Delta_i \in O' \setminus O$ and $\partial\Delta_i \cap \partial O \neq \emptyset$, Δ_i meets the assumptions of Lemma 8 (thanks to the reductio ad absurdum). Thus Lemma 8 provides $\Delta_{i+1} \in O' \setminus O$ such that $\partial\Delta_{i+1} \cap \partial O \neq \emptyset$ and Δ_i is behind Δ_{i+1}. □

5.4 Proof of star-shape shelling

First we show in Theorem 9 that $c(O) \cap c(\Delta)$ is a 2-ball using the tetrahedron Δ provided by Theorem 8. The proof highly relies on the fact that we are not in a degenerate case, i.e. $\mathbf{c}$ is not in a plane including a triangle of $c(T)$. It also requires the three following lemmas. Then a shelling between O and O' is obtained in a corollary of Theorem 9.

Lemma 10 *If O is $\mathbf{c}$-star-shaped and Δ is defined by Theorem 8, $O \cup \{\Delta\}$ is also $\mathbf{c}$-star-shaped.*

Proof According to Theorem 7 applied to O, every triangle $t \in \partial O \setminus \partial\Delta$ is a face of a tetrahedron $\Delta' \in O$ such that Δ' and $\mathbf{c}$ are in the same t half-space. Theorem 8 implies that every triangle $t \in \partial\Delta \setminus \partial O$ is such that Δ and $\mathbf{c}$ are in the same t half-space. Thanks to Lemma 2, every triangle $t \in \partial(O \cup \{\Delta\})$ is a face of a tetrahedron $\Delta' \in O \cup \{\Delta\}$ such that Δ' and $\mathbf{c}$ are in the same t half-space. Last we apply Theorem 7 to $O \cup \{\Delta\}$ and see that $O \cup \{\Delta\}$ is $\mathbf{c}$-star-shaped. □

The proofs of the two following technical lemmas are in the supplementary material and Appendix B.

Lemma 11 *Let $\mathbf{abcd}$ be a tetrahedron and K be a simplicial complex such that $K \subseteq c(\mathbf{abcd})$. If $c(\mathbf{abc}) \subsetneq K$, $\mathbf{d} \in K$. If $c(\mathbf{abc}, \mathbf{bcd}) \subsetneq K$, $\mathbf{ad} \in K$.*

Lemma 12 *Let $\Delta = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ and triangles $t_i \in \partial\Delta$ such that $\mathbf{v}_i \notin t_i$. Let H_i and H'_i be the two half-spaces of t_i such that $\Delta \subset H_i$. We have $H'_1 \cap H'_2 \cap H'_3 \cap H'_4 = \emptyset$. If $\mathbf{c} \in H'_1 \cap H_2 \cap H_3 \cap H_4$, $\mathbf{v}_1\mathbf{c} \cap t_1 \neq \emptyset$. Let $\mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)/2$. If $\mathbf{c} \in H'_1 \cap H'_2 \cap H_3 \cap H_4$, $\mathbf{vc} \cap (t_1 \cup t_2) \neq \emptyset$ (Fig. 11).*

Theorem 9 *If O is $\mathbf{c}$-star-shaped and Δ is defined by Theorem 8, $c(\partial\Delta) \cap c(\partial O)$ is a 2-ball.*

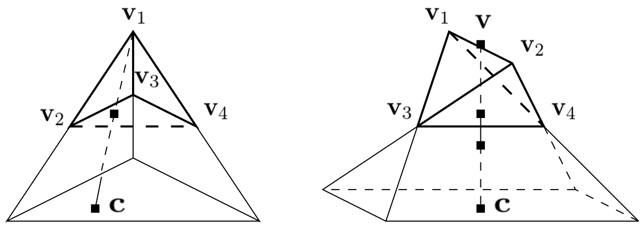

Fig. 11 The point $\mathbf{vc} \cap t_1$ according to Lemma 12. We have $t_1 = \mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$, $t_2 = \mathbf{v}_3\mathbf{v}_4\mathbf{v}_1$, $t_3 = \mathbf{v}_4\mathbf{v}_1\mathbf{v}_2$ and $t_4 = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$. Left: $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{c} \in H'_1 \cap H_2 \cap H_3 \cap H_4$. The half-spaces H'_1, H_2, H_3 and H_4 are below t_1, t_2, t_3 and t_4 respectively. Right: $\mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)/2$ and $\mathbf{c} \in H'_1 \cap H'_2 \cap H_3 \cap H_4$. The half-spaces H'_1, H'_2, H_3 and H_4 are below t_1, t_2, t_3 and t_4 respectively.

Proof Let triangles t_i be such that $\partial\Delta = \{t_1, t_2, t_3, t_4\}$ and $\partial\Delta \cap \partial O = \{t_1, \cdots, t_n\}$. Thanks to Theorem 8, $n \geq 1$. Let $K = c(\partial\Delta) \cap c(\partial O)$. We show that $K = c(t_1, \cdots, t_n)$ where $1 \leq n \leq 3$ and conclude using Theorem 3. Note that $c(t_1, \cdots, t_n) = c(\partial\Delta \cap \partial O) \subseteq K$.

Let H_i and H'_i be the two half-spaces of t_i such that $\Delta \subset H_i$. First we show that $i \leq n$ implies $\mathbf{c} \in H'_i$ and $i > n$ implies $\mathbf{c} \in H_i$. If $i \leq n$, there is a tetrahedron $\Delta_i \in O$ such that $t_i = \Delta \cap \Delta_i$ and $\Delta_i \subset H'_i$. Since O is $\mathbf{c}$-star-shaped, $\mathbf{c}$ and Δ_i are in the same t_i half-space (Theorem 7), i.e. $\mathbf{c} \in H'_i$. If $i > n$, $t_i \in \partial\Delta \setminus \partial O$. Thanks to Lemma 2, $t_i \in \partial(O \cup \{\Delta\})$. Since $\partial(O \cup \{\Delta\})$ is $\mathbf{c}$-star-shaped (Lemma 10), $\mathbf{c}$ and Δ are in the same t_i half-space (Theorem 7), i.e. $\mathbf{c} \in H_i$.

Second we show that $n \neq 4$. If $n = 4$ (reductio ad absurdum), $\mathbf{c} \in H'_1 \cap H'_2 \cap H'_3 \cap H'_4$, which is impossible thanks to Lemma 12.

Third we show that $n = 3$ implies $K = c(t_1, t_2, t_3)$. We have $c(t_1, t_2, t_3) \subseteq K \subseteq c(\partial\Delta) = c(t_1, t_2, t_3) \cup \{t_4\}$. If $c(t_1, t_2, t_3) \subsetneq K$ (reductio ad absurdum), $K = c(\partial\Delta)$, i.e. $n = 4$ (impossible).

Fourth we show that $n = 1$ implies $K = c(t_1)$. Thus $\mathbf{c} \in H'_1 \cap H_2 \cap H_3 \cap H_4$. Let $\mathbf{v}_1$ be the Δ vertex that is not in t_1, i.e. $\mathbf{v}_1 = t_2 \cap t_3 \cap t_4$. Thanks to Lemma 12, there is a point $\mathbf{w} \in t_1 \cap \mathbf{cv}_1$. Assume (reductio ad absurdum) that $c(t_1) \subsetneq K$. According to Lemma 11, $\mathbf{v}_1 \in K \subseteq c(O)$. Since O is $\mathbf{c}$-star-shaped, $\mathbf{cv}_1 \subset |O|$. We obtain $\mathbf{wv}_1 \subseteq \Delta \cap |O|$. Since $\Delta \notin O$, Lemma 5 implies $\mathring{\Delta} \cap |O| = \emptyset$. Therefore $\mathbf{wv}_1 \subset |\partial\Delta|$. There is a triangle $t_i \in \partial\Delta$ that includes at least two distinct points in $\mathbf{wv}_1$. Since $\mathbf{c} \in (\mathbf{wv}_1)$, $\mathbf{c}$ is in the t_i plane: we are in a degenerate case (contradiction).

Last we show that $n = 2$ implies $K = c(t_1, t_2)$. Thus $\mathbf{c} \in H'_1 \cap H'_2 \cap H_3 \cap H_4$. Let $\mathbf{v}$ be the center of the edge $t_3 \cap t_4$. Thanks to Lemma 12, there is a point $\mathbf{w} \in (t_1 \cup t_2) \cap \mathbf{cv}$. Assume (reductio ad absurdum) that $c(t_1, t_2) \subsetneq K$. According to Lemma 11, $t_3 \cap t_4 \in K \subseteq c(O)$ and $\mathbf{v} \in |O|$. Since O is $\mathbf{c}$-star-shaped, $\mathbf{cv} \subset |O|$.

Therefore $\mathbf{wv} \subseteq \Delta \cap |O|$. Last we show that $\mathbf{c}$ is in a t_i plane as in the previous case $n = 1$ (contradiction). □

Now we obtain the result of Sec. 5.

Corollary 3 *Let $\mathbf{c}$ be a point in $|T|$ that is not in a plane including a triangle of $c(T)$. Let $O \subseteq T$ such that O is non-empty $\mathbf{c}$-star-shaped. Then ∂O is a 2-manifold. Let $O' \subseteq T$ such that $O \subsetneq O'$ and O' is $\mathbf{c}$-star-shaped. Then there is a shelling $O_0 \cdots O_n$ such that $O_0 = O$, $O_n = O'$ and every $O_i \subseteq T$ is $\mathbf{c}$-star-shaped.*

Proof First we assume that both $\tilde{O}$ and O_i are $\mathbf{c}$-star-shaped such that $\emptyset \neq O_i \subsetneq \tilde{O} \subseteq T$ and ∂O_i is a 2-manifold. We show that there is a tetrahedron $\Delta_i \in \tilde{O} \setminus O_i$ such that $\partial \Delta_i \cap \partial O_i \neq \emptyset$ and, using notation $O_{i+1} = O_i \cup \{\Delta_i\}$, O_{i+1} is $\mathbf{c}$-star-shaped and ∂O_{i+1} is a 2-manifold. Theorem 8 provides the tetrahedron Δ_i. Let $O_{i+1} = O_i \cup \{\Delta_i\}$. Now O_{i+1} is $\mathbf{c}$-star-shaped (Lemma 10) and $c(\partial \Delta_i) \cap c(\partial O_i)$ is a 2-ball (Theorem 9). Thus $c(\Delta_i) \cap c(O_i)$ is non-empty 2D-pure (Theorem 3) and ∂O_{i+1} is a 2-manifold (Theorem 1).

Second we show that ∂O is a 2-manifold. Let Δ' be a tetrahedron such that $\mathbf{c} \in \Delta' \in T$. Let $O_0 = \{\Delta'\}$. Note that O_0 is $\mathbf{c}$-star-shaped (Δ' is convex) and ∂O_0 is a 2-manifold (∂O_0 is a 2-sphere). By successive use(s) of the first step (above) with $\tilde{O} = O$, we find a series $O_0 \cdots O_n$ in T such that $O_n = O$ and every ∂O_i is a 2-manifold.

Third we show that there is a shelling $O_0 \cdots O_n$ such that $O_0 = O$, $O_n = O'$ and every O_i is $\mathbf{c}$-star-shaped. We set $O_0 = O$. Note that O_0 is $\mathbf{c}$-star-shaped and ∂O_0 is a 2-manifold. By successive use(s) of the first step (above) with $\tilde{O} = O'$, we find a series $O_0 \cdots O_n$ in T such that $O_n = O'$ and every O_i is $\mathbf{c}$-star-shaped. This series is a shelling since $O_{i+1} = O_i \cup \{\Delta_i\}$ and $c(\Delta_i) \cap c(O_i)$ is non-empty 2D-pure for all $i < n$, and ∂O_0 is a non-empty 2-manifold. □

According to Appendix C, there are non-shellable star-shapes in degenerate cases.

5.5 A qualitative argument for surface reconstruction

Here we provide a qualitative argument (this is not a proof as in the paper remainder) to explain that the first shelling in [13], i.e. Algorithm 1 started from $O = \emptyset$, does not have excessive blocking and provides most of the tetrahedra enclosed by the final surface. Sec. 3.1 gives a reminder of this algorithm and useful notations F and n_Δ.

The set F looks like a star-shape: for every point $\mathbf{x}$ reconstructed from a camera location $\mathbf{c}$, $\mathbf{cx} \subset \cup F$ (F is not exactly a star-shape since $\mathbf{x}$ is restricted to the vertices in $c(T)$ and there are several $\mathbf{c}$). Assuming that $\mathbf{c}$ follows a curve and the distance between $\mathbf{x}$ and $\mathbf{c}$ is small compared to the curve length, F can be seen as a tubular neighborhood of the curve.

Furthermore, O also looks like a star-shape during the first shelling in [13]. Since both Δ size and $\mathbf{cx}$ density increase when we go toward the curve, n_Δ increases when we go toward the curve. Since this shelling tries to add first to O the tetrahedra Δ having the highest n_Δ, O essentially grows in a front-to-back order relatively to the curve. As a consequence, the points in $\cup F$ with a small distance to the curve enter in $\cup O$ during the shelling before those with a greater distance, i.e. there is a $\mathbf{c}$ in the curve such that $\mathbf{cx} \subset \cup O$ if $\mathbf{x} \in \cup O$.

Now O is growing in F by the shelling such that both O and F looks like star-shapes with the same curve as "center". Then Corollary 3 suggests that O fills F, if we replace the point center by the curve center.

Let O/F be the ratio of the number of the tetrahedra in O generated by the first shelling alone and the number of the tetrahedra in F. Let O/O_e be the ratio of the number of the tetrahedra in O generated by the first shelling alone and the final number of the tetrahedra generated by all operators (including the first shelling). In Tab. 2 of [13], $O/F = 87\%$ and $O/O_e = 94\%$.

6 Shelling blocking

In this section, we provide a family of 3-balls that are strongly non-shellable with small numbers of simplices (Sec. 6.5). We also provide other cases of blocking (Sec. 6.6) for a shelling started from a general $O_0 \subseteq T$. All these cases of shelling blocking are generated by disjoint unions of tetrahedron sets in T that are called "pipe" (Sec. 6.1) and "slice" (Sec. 6.3). Both are 3-balls and their "gluing" (union of such a 3-ball with another tetrahedron set in T^∞) are described in Secs. 6.2 and 6.4.

6.1 Basic component: pipe

If triangles $\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0$ and $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1$ have distinct vertices, the tetrahedron set $P = p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ is

$$\{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0, \mathbf{a}_0\mathbf{a}_1\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1, \mathbf{a}_0\mathbf{b}_1\mathbf{c}_0\mathbf{c}_1\} \quad (14)$$

We assume that $P \subseteq T$, show P in Fig. 12, and see that $\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0$ is an internal tetrahedron of P. Indeed, every triangle face of $\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0$ is join to $\mathbf{b}_0$ or $\mathbf{c}_1$ by another tetrahedron in P:

$$P = \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0\} \cup \mathbf{b}_0 \times \{\mathbf{a}_0\mathbf{a}_1\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_0\} \cup \mathbf{c}_1 \times \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1, \mathbf{a}_0\mathbf{b}_1\mathbf{c}_0\}. \quad (15)$$

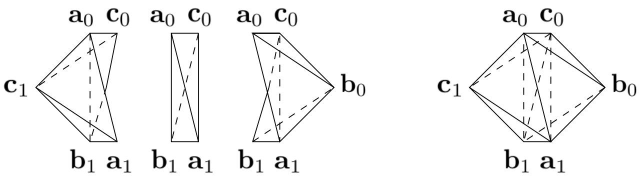

Fig. 12 The pipe $p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ (right) and its decomposition (left). See the internal tetrahedron $\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0$.

Lemma 13 *We have*

$$\partial P = \{\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1\} \cup \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_0, \mathbf{a}_1\mathbf{b}_0\mathbf{b}_1\} \cup \{\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1\} \cup \{\mathbf{a}_0\mathbf{c}_0\mathbf{c}_1, \mathbf{a}_0\mathbf{a}_1\mathbf{c}_1\}. \quad (16)$$

Proof Eq. 15 implies that the triangles in $c(P)$, that have neither $\mathbf{b}_0$ nor $\mathbf{c}_1$ as vertex, are faces of the internal tetrahedron $\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0$. Since $\partial P \subset c(P)$, every triangle in ∂P has vertex $\mathbf{b}_0$ or $\mathbf{c}_1$, i.e.

$$\partial P \subseteq \mathbf{b}_0 \times \mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{c}_0\text{-}\mathbf{a}_0 \cup \mathbf{b}_0 \times \mathbf{a}_1\text{-}\mathbf{b}_1\text{-}\mathbf{c}_0\text{-}\mathbf{a}_1 \cup \mathbf{c}_1 \times \mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{b}_1\text{-}\mathbf{a}_0 \cup \mathbf{c}_1 \times \mathbf{a}_0\text{-}\mathbf{b}_1\text{-}\mathbf{c}_0\text{-}\mathbf{a}_0. \quad (17)$$

Since a triangle in ∂P is in a single tetrahedron of P,

$$\begin{aligned}\partial P &= \mathbf{b}_0 \times \mathbf{c}_0\text{-}\mathbf{a}_0\text{-}\mathbf{a}_1 \cup \mathbf{b}_0 \times \mathbf{a}_1\text{-}\mathbf{b}_1\text{-}\mathbf{c}_0 \cup \mathbf{c}_1 \times \mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{b}_1 \cup \mathbf{c}_1 \times \mathbf{b}_1\text{-}\mathbf{c}_0\text{-}\mathbf{a}_0 \\ &= \mathbf{b}_0 \times \mathbf{c}_0\text{-}\mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{b}_1\text{-}\mathbf{c}_0 \cup \mathbf{c}_1 \times \mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{b}_1\text{-}\mathbf{c}_0\text{-}\mathbf{a}_0 \quad (18)\end{aligned}$$

and we obtain the result. □

We say that P is a *pipe*. Intuitively, the two cross sections of the pipe are triangles $\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0$ and $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1$. The lateral section of the pipe is a strip of 6 triangles (in Eq. 16) having edges $\mathbf{a}_0\mathbf{a}_1$, $\mathbf{b}_0\mathbf{b}_1$, $\mathbf{c}_0\mathbf{c}_1$ and three others. A pipe series $p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$, $p(\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1, \mathbf{a}_2\mathbf{b}_2\mathbf{c}_2)$, $p(\mathbf{a}_2\mathbf{b}_2\mathbf{c}_2, \mathbf{a}_3\mathbf{b}_3\mathbf{c}_3)$, $\cdots$ will be used to form a longer "pipe" and build strongly non-shellable 3-balls.

Lemma 14 *If* $p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1) \subseteq T$*,* P *is a 3-ball.*

The proof ot this lemma is in the supplementary material (use a shelling, or see that ∂P is an octahedron).

6.2 Gluing a pipe

The goal of Sec. 6.2 is Theorem 11: we define an union of a pipe P and another tetrahedron set $A \subseteq T^\infty$ such that the criterion of the strongly non-shellability (Sec. 4.1) applied to $A \cup P$ is meet for every tetrahedron in P. The proof of this theorem (and others) needs the following gluing lemma and theorem.

Lemma 15 *If* $A \subseteq T^\infty$ *and* $B \subseteq T^\infty$ *and* $A \cap B = \emptyset$*,* $\partial(A \cup B) = (\partial A \setminus \partial B) \cup (\partial B \setminus \partial A)$*.*

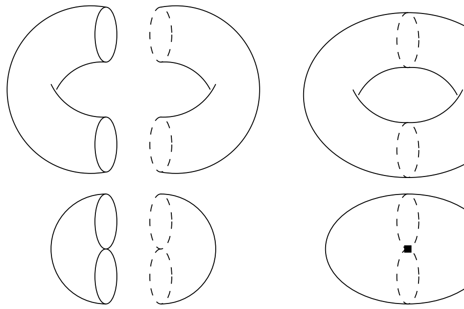

Fig. 13 Gluing $A \subset T$ (left) and $B \subset T$ (middle) using Theorem 10. Both ∂A and ∂B are 2-manifolds, $A \cap B = \emptyset$ and $\partial A \cap \partial B$ is the union of two 2-balls. Right: the result $A \cup B$. If $c(A) \cap c(B)$ is a 2-manifold with boundary, $\partial(A \cup B)$ is a 2-manifold (top). If $c(A) \cap c(B)$ is 2D pure and non-manifold, $\partial(A \cup B)$ can have a singular vertex (bottom).

Proof Let[1] $t \in c(T^\infty)$ and $C = T^\infty \setminus (A \cup B)$. Let Δ and Δ' be the two tetrahedra in T^∞ such that $t = \Delta \cap \Delta'$. We show that $t \in \partial(A \cup B)$ iff $t \in (\partial A \setminus \partial B) \cup (\partial B \setminus \partial A)$ in all cases.

If $(\Delta, \Delta') \in A \times A$, the triangle t is neither in $\partial A \setminus \partial B$ nor in $\partial(A \cup B)$. Since $A \cap B = \emptyset$, we also have $t \notin \partial B \setminus \partial A$. The case $(\Delta, \Delta') \in B \times B$ is similar. If $(\Delta, \Delta') \in C \times C$, t is neither in $\partial(A \cup B)$ nor in $(\partial A \setminus \partial B) \cup (\partial B \setminus \partial A)$. If $(\Delta, \Delta') \in A \times B$, $t \notin \partial(A \cup B)$. Since $A \cap B = \emptyset$, $t \in \partial A \cap \partial B$ and thus $t \notin (\partial A \setminus \partial B) \cup (\partial B \setminus \partial A)$. If $(\Delta, \Delta') \in A \times C$, $t \in \partial A \setminus \partial B$ and $t \in \partial(A \cup B)$. The case $(\Delta, \Delta') \in B \times C$ is similar. □

Theorem 10 *Let* $A \subseteq T^\infty$ *and* $B \subseteq T \setminus A$ *such that both* ∂A *and* ∂B *are 2-manifolds, and* $c(A) \cap c(B)$ *is a 2-manifold with boundary. Then* $\partial(A \cup B)$ *is a 2-manifold.*

This theorem means that volumes A and B can be glued properly, i.e. by maintaining the 2-manifold property of their boundaries, if their intersection is well behaved (Fig. 13). The proof is technical and is in Appendix D.

Theorem 11 *Let* $A \subseteq T^\infty$ *such that* ∂A *is a 2-manifold. Let* $P = p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ *such that* $P \subseteq T \setminus A$ *and*

$$c(A) \cap c(P) = c(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1, \mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1). \quad (19)$$

Then $\partial(A \cup P)$ *is a 2-manifold. If a tetrahedron* $\Delta \in P$*,* $c(\partial(A \cup P)) \cap c(\Delta)$ *is not a 2-ball (Fig. 14).*

Proof Since the triangles in $c(A) \cap c(P)$ form a hexagon $\mathbf{a}_0\mathbf{b}_0\mathbf{b}_1\mathbf{a}_1\mathbf{c}_1\mathbf{c}_0$, $c(A) \cap c(P)$ is homeomorphic to a 2-ball. Then Lemma 14 and Theorem 10 imply that $\partial(A \cup P)$ is a 2-manifold.

We will show that $c(\partial(A \cup P)) \cap c(\Delta)$ has zero or one triangle and includes four vertices if $\Delta \in P$. This

[1] Addendum: t is a triangle.

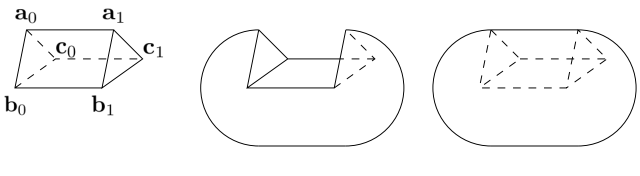

Fig. 14 Assumptions of Theorem 11: P (left), A (middle) and $A \cup P$ (right).

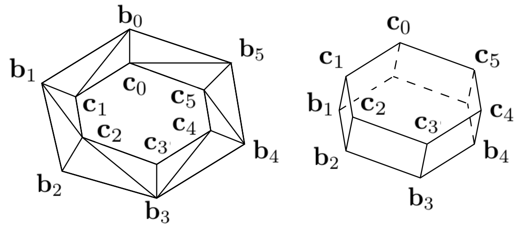

Fig. 15 Annulus (left) and slice (right) using $n = 6$.

implies that $c(\partial(A \cup P)) \cap c(\Delta)$ is not 2D pure, which in turn implies that $c(\partial(A \cup P)) \cap c(\Delta)$ is not a 2-ball thanks to Theorem 3.

Since $\partial A \cap \partial P = \{\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1, \mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1\}$, $\partial P \setminus \partial A = \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_0, \mathbf{a}_1\mathbf{b}_0\mathbf{b}_1, \mathbf{a}_0\mathbf{c}_0\mathbf{c}_1, \mathbf{a}_0\mathbf{a}_1\mathbf{c}_1\}$ (Eq. 16). Therefore

$$\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1, \mathbf{c}_0, \mathbf{c}_1\} \subset c(\partial P \setminus \partial A) \tag{20}$$

and

$$\begin{aligned}
&\Delta = \mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0 \Rightarrow \partial\Delta \cap (\partial P \setminus \partial A) = \emptyset \\
&\Delta = \mathbf{a}_0\mathbf{a}_1\mathbf{b}_0\mathbf{c}_0 \Rightarrow \partial\Delta \cap (\partial P \setminus \partial A) = \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_0\} \\
&\Delta = \mathbf{a}_1\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0 \Rightarrow \partial\Delta \cap (\partial P \setminus \partial A) = \{\mathbf{a}_1\mathbf{b}_0\mathbf{b}_1\} \\
&\Delta = \mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1 \Rightarrow \partial\Delta \cap (\partial P \setminus \partial A) = \{\mathbf{a}_0\mathbf{a}_1\mathbf{c}_1\} \\
&\Delta = \mathbf{a}_0\mathbf{b}_1\mathbf{c}_0\mathbf{c}_1 \Rightarrow \partial\Delta \cap (\partial P \setminus \partial A) = \{\mathbf{a}_0\mathbf{c}_0\mathbf{c}_1\}.
\end{aligned} \tag{21}$$

Since $\Delta \in P$ has its four vertices in $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{b}_0, \mathbf{b}_1, \mathbf{c}_0, \mathbf{c}_1\}$ and thanks to Eq. 20, $c(\partial P \setminus \partial A) \cap c(\Delta)$ includes four vertices. Furthermore, Eq. 21 implies that $(\partial P \setminus \partial A) \cap (\partial\Delta)$ has zero or one triangle for every $\Delta \in P$.

Assume (reductio ad absurdum) that there is a triangle $t \in (\partial A \setminus \partial P) \cap \partial\Delta$. Thus $t \subseteq \Delta \in P$ and $t \subseteq \Delta' \in A$. Since $A \cap P = \emptyset$ (Eq. 19), we have $\Delta' \notin P$ and obtain $t \in \partial P$, which contradicts $t \in \partial A \setminus \partial P$.

Thanks to Lemma 15, $(\partial P \setminus \partial A) \cup (\partial A \setminus \partial P) = \partial(A \cup P)$. Therefore $c(\partial(A \cup P))$ includes the four vertices of Δ and has zero or one triangle of Δ. □

6.3 Basic component: slice

Let integer $n \geq 3$ and vertices $\mathbf{b}_0, \cdots \mathbf{b}_n$ and $\mathbf{c}_0, \cdots \mathbf{c}_n$ in $c(T)$ that are distinct, with exceptions $\mathbf{b}_n = \mathbf{b}_0$ and $\mathbf{c}_n = \mathbf{c}_0$. We have cycles $\mathbf{b}_* = \mathbf{b}_0\text{-}\mathbf{b}_1\text{-}\cdots\mathbf{b}_{n-1}\text{-}\mathbf{b}_0$ and $\mathbf{c}_* = \mathbf{c}_0\text{-}\mathbf{c}_1\text{-}\cdots\mathbf{c}_{n-1}\text{-}\mathbf{c}_0$. Let

$$\begin{aligned}
&N = \bigcup_{0 \leq i < n} N_i \text{ where} \\
&N_i = \{\mathbf{b}_i\mathbf{b}_{i+1}\mathbf{c}_i, \mathbf{b}_{i+1}\mathbf{c}_i\mathbf{c}_{i+1}\} \text{ or} \\
&N_i = \{\mathbf{c}_{i+1}\mathbf{b}_i\mathbf{b}_{i+1}, \mathbf{c}_i\mathbf{c}_{i+1}\mathbf{b}_i\}.
\end{aligned} \tag{22}$$

The set N is a strip of triangles that form an *annulus* and we have $\partial N = \mathbf{b}_* \cup \mathbf{c}_*$. We say that $S \subseteq T$ is a *slice* if S is a 3-ball such that $N \subseteq \partial S$ (see Fig. 15).

Slice properties are in the following theorem:

Theorem 12 *Let S be a slice with an annulus $N \subseteq \partial S$. There are B and C such that $\partial S = N \cup B \cup C$, $\partial B = \mathbf{b}_*$, $\partial C = \mathbf{c}_*$, $c(B) \cap c(C) = \emptyset$, $c(B) \cap c(N) = c(\mathbf{b}_*)$, $c(C) \cap c(N) = c(\mathbf{c}_*)$, both B and C are strongly connected.*

Intuitively, the boundary of the 3-ball S is a 2-sphere segmented by N in three connected parts B, N, C, whose common boundaries are $\mathbf{b}_*$ or $\mathbf{c}_*$. Furthermore, S has two "sides" $|B|$ and $|C|$ which do not intersect. In the right of Fig. 15, B is on the bottom and C is on the top of the slice. The proof is technical (see Appendix E).

6.4 Gluing a slice

Similarly as in Sec. 6.2, the goal of Sec. 6.4 is Corollary 4: we define an union of a slice S' and another tetrahedron set $A \subseteq T^\infty$ such that the criterion of the strongly non-shellability applied to $A \cup S'$ is meet for every tetrahedron in S'. Here S' is computed by "shrinking" an initial slice S: we progressively remove every tetrahedron Δ from S that does not meet this criterion. The proof needs Theorem 13, which guarantees that S' is a slice (a 3-ball with the same annulus as S). This theorem needs the following lemma.

Lemma 16 *Let $A \subseteq T^\infty$ such that ∂A is a 2-manifold. Let $S \subset T \setminus A$ be a 3-ball such that $c(A) \cap c(S) = c(N)$ and N is an annulus. Then there are B and C as in Theorem 12 such that $c(\partial(A \cup S)) \cap c(S) = c(B) \cup c(C)$.*

Proof Since $c(A) \cap c(S) = c(N)$ and $A \cap S = \emptyset$, every triangle in N is in a tetrahedron in A and another in S. Thus S is a slice with the annulus $N \subset \partial S$. Let B and C as in Theorem 12. Now $N = \partial A \cap \partial S$ and Lemma 15 imply $\partial(A \cup S) = (\partial A \setminus N) \cup (\partial S \setminus N)$. We use shortened notation $\bar{S} = c(S)$. Since $\partial(A \cup S) = (\partial A \setminus N) \cup (\partial S \setminus N)$ and $\partial S \setminus N = B \cup C$ and $B \subset \bar{S}$ and $C \subset \bar{S}$,

$$\begin{aligned}
c(\partial(A \cup S)) \cap \bar{S} &= (c(\partial A \setminus N) \cap \bar{S}) \cup (c(\partial S \setminus N) \cap \bar{S}) \\
&= (c(\partial A \setminus N) \cap \bar{S}) \cup c(B) \cup c(C).
\end{aligned} \tag{23}$$

Let σ be a simplex in $c(\partial A \setminus N) \cap \bar{S}$ and show that $\sigma \in c(B) \cup c(C)$. Since $c(\partial A \setminus N) \cap \bar{S} \subseteq c(A) \cap \bar{S} = c(N)$, we have $\sigma \in c(N)$. We have $c(N) = c(\mathbf{b}_*) \cup c(\mathbf{c}_*) \cup \tilde{N}$ using

$$\begin{aligned}&\tilde{N} = \cup_i \tilde{N}_i \text{ where}\\ &\tilde{N}_i = \{\mathbf{b}_i\mathbf{c}_i, \mathbf{b}_{i+1}\mathbf{c}_{i+1}, \mathbf{b}_{i+1}\mathbf{c}_i, \mathbf{b}_i\mathbf{b}_{i+1}\mathbf{c}_i, \mathbf{b}_{i+1}\mathbf{c}_i\mathbf{c}_{i+1}\} \text{ or}\\ &\tilde{N}_i = \{\mathbf{b}_i\mathbf{c}_i, \mathbf{b}_{i+1}\mathbf{c}_{i+1}, \mathbf{b}_i\mathbf{c}_{i+1}, \mathbf{c}_{i+1}\mathbf{b}_i\mathbf{b}_{i+1}, \mathbf{c}_i\mathbf{c}_{i+1}\mathbf{b}_i\}\end{aligned} \quad (24)$$

If $\sigma \in c(\mathbf{b}_*) \cup c(\mathbf{c}_*)$, $\sigma \in c(B) \cup c(C)$. Otherwise $\sigma \in \tilde{N}$. This implies that there is an edge $\mathbf{b}_i\mathbf{c}_i$ or $\mathbf{b}_{i+1}\mathbf{c}_i$ or $\mathbf{c}_{i+1}\mathbf{b}_i$ in $c(\partial(A \cup S)) \cap c(N)$. Let e be this edge. There are distinct triangles t_1 and t_2 in $N = \partial S \cap \partial A$ such that $e = t_1 \cap t_2$. Since $e \in c(\partial(A \cup S)) = c((\partial A \cup \partial S) \setminus N)$, there is another triangle $t_3 \in (\partial A \cup \partial S) \setminus N$ such that $e \subset t_3$. If $t_3 \in \partial A$, e is in 3 distinct triangles in ∂A. This is impossible since ∂A is a 2-manifold. If $t_3 \in \partial S$, e is in 3 distinct triangles in ∂S. This is impossible since ∂S is a 2-manifold. □

Theorem 13 *Let $A \subseteq T^\infty$ such that ∂A is a 2-manifold. Let $S \subset T \setminus A$ be a 3-ball such that $c(A) \cap c(S) = c(N)$ and N is an annulus. Let Δ be a tetrahedron in S such that $c(\partial(A \cup S)) \cap c(\Delta)$ is a 2-ball. Then $S \setminus \{\Delta\}$ is a 3-ball and $c(A) \cap c(S \setminus \{\Delta\}) = c(N)$.*

Proof Let B and C be as in Lemma 16. Let $X = c(\Delta) \cap c(B)$ and $Y = c(\Delta) \cap c(C)$. First we show that $X = \emptyset$ (or similarly, $Y = \emptyset$). Since $\Delta \in S$,

$$\begin{aligned}c(\Delta) \cap c(\partial(A \cup S)) &= c(\Delta) \cap (c(\partial(A \cup S)) \cap c(S))\\ &= X \cup Y.\end{aligned} \quad (25)$$

Assume (reductio ad absurdum) that $X \neq \emptyset$ and $Y \neq \emptyset$. Since $c(B) \cap c(C) = \emptyset$, $X \cap Y = \emptyset$ and $c(\Delta) \cap c(\partial(A \cup S))$ is not connected. This contradicts that $c(\Delta) \cap c(\partial(A \cup S))$ is a 2-ball.

Second we show that $c(\Delta) \cap c(N) \subseteq c(\Delta) \cap c(\mathbf{c}_*)$. Since $c(\Delta) \cap c(\mathbf{b}_*) \subseteq X = \emptyset$, Δ does not have a vertex in $c(\mathbf{b}_*)$. Since every vertex in $c(\Delta) \cap c(N)$ is a $\mathbf{b}_i$ or a $\mathbf{c}_i$, every vertex in $c(\Delta) \cap c(N)$ is a $\mathbf{c}_i$. Thus $c(\Delta) \cap c(N) \subseteq c(\mathbf{c}_*)$. We obtain $c(\Delta) \cap c(N) \subseteq c(\Delta) \cap c(\mathbf{c}_*)$.

Third we show that $S \setminus \{\Delta\}$ is a 3-ball. Since $\partial S = B \cup C \cup N$ and $c(\Delta) \cap c(N) \subseteq c(\Delta) \cap c(\mathbf{c}_*) \subseteq c(\Delta) \cap c(C) = Y$,

$$\begin{aligned}c(\Delta) \cap c(\partial S) &= X \cup Y \cup (c(\Delta) \cap c(N)) = Y\\ &= c(\Delta) \cap c(\partial(A \cup S)).\end{aligned} \quad (26)$$

Therefore $c(\Delta) \cap c(\partial S)$ is a 2-ball. Since S is a 3-ball, we see that $S \setminus \{\Delta\}$ is a 3-ball (use Corollary 2 for the shelling $O_0 = T^\infty \setminus S$, $O_1 = O_0 \cup \{\Delta\}$ and Theorem 5).

Last we show that $c(A) \cap c(S) = c(A) \cap c(S \setminus \{\Delta\})$. We have

$$c(A) \cap c(S) = (c(A) \cap c(S \setminus \{\Delta\})) \cup (c(A) \cap c(\Delta)). \quad (27)$$

Since $\Delta \in S$ and $c(N) = c(A) \cap c(S)$,

$$c(A) \cap c(\Delta) = c(N) \cap c(\Delta) \subseteq c(\Delta) \cap c(\mathbf{c}_*). \quad (28)$$

Let σ be a simplex in $c(A) \cap c(\Delta)$. Thanks to Eq. 28, σ is a vertex or an edge. Since $c(N) = c(A) \cap c(S)$ and thanks to Eq. 27, σ is in a triangle $t \in N$. Since $c(A) \cap c(\Delta)$ does not have a triangle, $t \in c(A) \cap c(S \setminus \{\Delta\})$. Thus $c(A) \cap c(\Delta) \subseteq c(A) \cap c(S \setminus \{\Delta\})$. □

Corollary 4 *Let $A \subseteq T^\infty$ such that ∂A is a 2-manifold. Let $S \subset T \setminus A$ be a 3-ball such that $c(A) \cap c(S) = c(N)$ and N is an annulus. There is a 3-ball $S' \subseteq S$ such that $c(A) \cap c(S') = c(N)$ and $c(\Delta) \cap c(\partial(A \cup S'))$ is not a 2-ball for every tetrahedron $\Delta \in S'$.*

Proof Let $S' = S$. We consider the following loop-algorithm: while there is a tetrahedron $\Delta \in S'$ such that $c(\Delta) \cap c(\partial(A \cup S'))$ is a 2-ball, remove Δ from S'. Thanks to Theorem 13, S' is always a 3-ball and we always have $c(A) \cap c(S') = c(N)$. The algorithm stops since S is finite, and it stops if $c(\Delta) \cap c(\partial(A \cup S'))$ is not a 2-ball for every tetrahedron $\Delta \in S'$. □

6.5 Family of strongly non-shellable 3-balls

First we need a specialization of Theorem 10 (proof in Appendix F):

Theorem 14 *Let $A \subseteq T^\infty$ and $B \subseteq T \setminus A$ such that ∂A is a connected 2-manifold, ∂B is a 2-sphere, $c(A) \cap c(B)$ is a 2-ball. Then $\partial(A \cup B)$ is homeomorphic to ∂A.*

Then Theorem 15 presents a family of strongly non-shellable 3-balls.

Theorem 15 *Let $P_i = p(\mathbf{a}_i\mathbf{b}_i\mathbf{c}_i, \mathbf{a}_{i+1}\mathbf{b}_{i+1}\mathbf{c}_{i+1})$ or $P_i = p(\mathbf{a}_{i+1}\mathbf{b}_{i+1}\mathbf{c}_{i+1}, \mathbf{a}_i\mathbf{b}_i\mathbf{c}_i)$ such that $P_i \subseteq T$ if $0 \leq i < n$. Let $S \subseteq T \setminus \cup_i P_i$ be a 3-ball such that $c(\cup_i P_i) \cap c(S) = c(N)$ and N is an annulus using notations in Eq. 22 (top of Fig. 16). Then there is a 3-ball $S' \subseteq S$ such that $c(\cup_i P_i) \cap c(S') = c(N)$ and $\cup_i P_i \cup S'$ is a strongly non-shellable 3-ball (bottom of Fig. 16).*

Intuitively (Fig. 16), $\partial(\cup_i P_i)$ is a torus whose intersection with ∂S (and also $\partial S'$) is the annulus N. Furthermore, $\partial(\cup_i P_i \cup S')$ is a 2-sphere, with both "sides" separated by the cycle $\mathbf{a}_*$ (B is in one side and C is in the other). If a tetrahedron Δ having a triangle in one side is removed from $\cup_i P_i \cup S'$, a singular vertex appears in both sides. We use Theorem 11 if $\Delta \in P_i$ and Corollary 4 if $\Delta \in S'$ to show that $c(\Delta) \cap c(\partial(\cup_i P_i \cup S'))$ is not a 2-ball, i.e. to show that $\cup_i P_i \cup S'$ is a strongly non-shellable 3-ball.

Proof First we find S'. Every ∂P_i is a 2-manifold and every $c(\partial(\cup_{k=0}^{i-1} P_k)) \cap c(\partial P_i)$ is a 2-manifold with boundary ($\{\mathbf{a}_i\mathbf{b}_i\mathbf{c}_i\}$ if $i < n-1$, $\{\mathbf{a}_{n-1}\mathbf{b}_{n-1}\mathbf{c}_{n-1}, \mathbf{a}_0\mathbf{b}_0\mathbf{c}_0\}$ if $i = n-1$). Thus successive uses of Theorem 10 imply that $\partial(\cup_{i=0}^{n-1} P_i)$ is a 2-manifold. Now we use Corollary 4

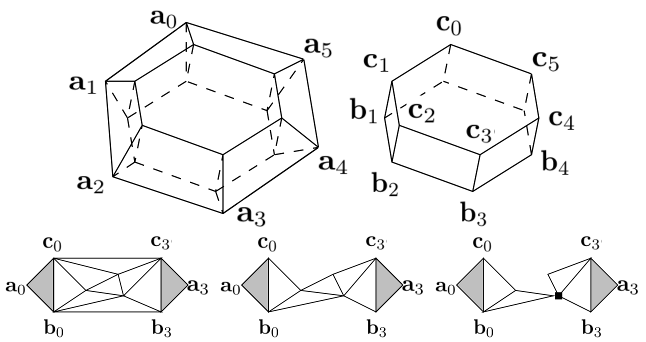

Fig. 16 Illustrations for Theorem 15. Top: S (right) and $\cup_i P_i \cup S$ (left). The pipes P_i form a torus; we have $c(\cup_i P_i) \cap c(S) = c(N)$ where N is the annulus whose boundary is $\mathbf{b}_* \cup \mathbf{c}_*$. Bottom: cross section by a plane including grey triangles $\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0$ and $\mathbf{a}_3\mathbf{b}_3\mathbf{c}_3$. The white triangles are tetrahedra in S (left), S' is obtained by removing some of these tetrahedra such that $c(\cup_i P_i) \cap c(S') = c(N)$ (middle), a singular vertex appears if we remove a tetrahedron from S' (right).

using $A = \cup_i P_i$ and obtain a 3-ball $S' \subseteq S$ such that $c(A) \cap c(S') = c(N)$ and $c(\Delta) \cap c(\partial(A \cup S'))$ is not a 2-ball for every tetrahedron $\Delta \in S'$.

Second we show that $A_i = \cup_{j \neq i} P_j \cup S'$ and $O = \cup_j P_j \cup S'$ are 3-balls. By successive uses of Theorem 14 (as in the previous case above), $\partial(\cup_{j \neq i} P_j)$ is a 2-manifold that is homeomorphic to a 2-sphere. Then Theorem 14 implies that ∂A_i is a 2-sphere (use $A = \cup_{j \neq i} P_j$ and $B = S'$). Now Theorem 14 implies that ∂O is a 2-sphere (use $A = A_i$ and $B = P_i$). Since $A_i \subseteq T$ and $O \subseteq T$, Theorem 5 implies that A_i and O are 3-balls.

Last we show that O is strongly non-shellable. We already know that $c(\Delta) \cap c(\partial O)$ is not a 2-ball for every tetrahedron $\Delta \in S'$. We use Theorem 11 using $A = A_i$ and $P = P_i$ and see that $c(\Delta) \cap c(\partial O)$ is not a 2-ball for every tetrahedron $\Delta \in P_i$. □

Appendix G shows that there is a strongly non-shellable 3-ball as described in Theorem 15: the strongly non-shellable 3-ball Z with 12 vertices and 25 tetrahedra in Sec. 4 of [24] (Z is the symmetric version of the 3-ball with 10 vertices and 21 tetrahedra).

6.6 Other shelling blocking

After a lemma, Theorem 16 provides examples of residual 3-balls in the free-space that cannot be carved by a shelling during the surface reconstruction.

Lemma 17 *If a 3-ball $O \subseteq T$ has at least two tetrahedra and $\Delta \in O$, there is a tetrahedron $\Delta' \in O \setminus \{\Delta\}$ such that $\Delta \cap \Delta'$ is a triangle.*

Proof Assume (reductio ad absurdum) that $\partial\Delta \subseteq \partial O$. Let $O_0 = T^\infty \setminus O$ and $O_1 = O_0 \cup \{\Delta\}$. We have $\Delta \in T \setminus O_0$ and $c(O_0) \cap c(\Delta) = c(\partial O) \cap c(\partial\Delta) = c(\partial\Delta)$ is non-empty 2D-pure. Thus (O_0, O_1) is a shelling. Since $O_1 = T^\infty \setminus (O \setminus \{\Delta\})$ and O has at least two tetrahedra, $O_1 \neq T^\infty$. Since ∂O_0 is a 2-sphere and $O_1 \neq T^\infty$, Corollary 2 implies that $c(O_0) \cap c(\Delta) = c(\partial\Delta)$ is a 2-ball (impossible). □

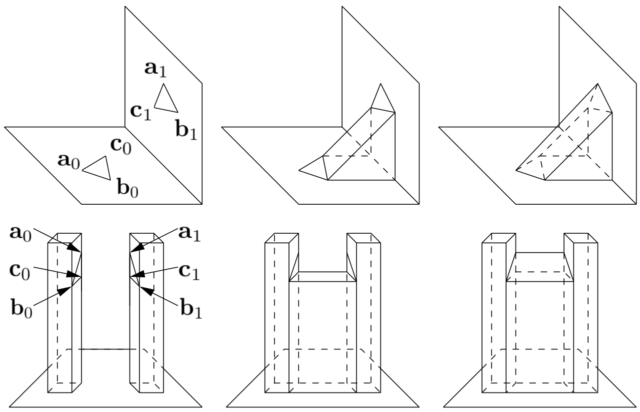

Fig. 17 Two examples for Theorem 16: M includes a horizontal ground and a vertical wall (top) or includes two pillars on the ground (bottom). Left: ∂M including triangles $\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0$ and $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1$. Middle: ∂M and S. Right: ∂M and S and P.

Theorem 16 *Let $M \subseteq T^\infty$ such that ∂M is a 2-manifold. Let $S \subseteq T \setminus M$ be a 3-ball such that $c(M) \cap c(S) = c(N \setminus \{\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1\})$ and N is an annulus (Eq. 22). Let $P = p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ such that $P \subseteq T \setminus (M \cup S)$ and $c(M) \cap c(P) = c(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ and $c(S) \cap c(P) = c(\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1)$. Then there is a 3-ball $S' \subseteq S$ such that $c(M \cup P) \cap c(S') = c(M \cup P) \cap c(S)$ and, using notation $O = T^\infty \setminus (M \cup P \cup S')$, ∂O is a 2-manifold, $c(O) \cap c(\Delta)$ is not 2D-pure for every tetrahedron $\Delta \in P \cup S'$ such that $\partial\Delta \cap \partial O \neq \emptyset$.*

Two examples of M, S and P are shown in Fig. 17. The first one (on the top) is inspired by the real example in Fig. 1 of [16] such that M is “the matter” (set of the tetrahedra non-intersected by rays and including the ground and walls of a city), P, S and $P \cup S$ are 3-balls that cannot be carved by a shelling although they are in the free-space (set of the tetrahedra intersected by rays). The second one (on the bottom) is similar and is inspired by the real example in Fig. 3.4 of [10].

Proof First we find S'. Let $A = M \cup P$. Since $c(M) \cap c(P)$ is two disjoint 2-balls and both ∂M and ∂P are 2-manifolds and $P \subset T \setminus M$, Theorem 10 implies that ∂A is a 2-manifold. Furthermore, $c(A) \cap c(S) = (c(M) \cap c(S)) \cup (c(P) \cap c(S)) = c(N)$. Thus Corollary 4 implies that there is a 3-ball $S' \subseteq S$ such that $c(A) \cap c(S') = c(N)$ and $c(\Delta) \cap c(\partial(A \cup S'))$ is not a 2-ball for every tetrahedron $\Delta \in S'$.

Second we show that ∂O is a 2-manifold. Since both ∂A and $\partial S'$ are 2-manifold and $c(A) \cap c(S')$ is a 2-manifold with boundary (the annulus N), Theorem 10 implies that $\partial(A \cup S')$ is a 2-manifold. Therefore $\partial O = \partial(M \cup P \cup S')$ is a 2-manifold.

Third we show that $\Delta \in P \cup S'$ implies that $c(\Delta) \cap c(\partial O)$ is not a 2-sphere. If $\Delta \in S'$ and since S' is a 3-ball with more that one tetrahedron, there is another tetrahedron $\Delta' \in S'$ such that $\Delta' \cap \Delta$ is a triangle (Lemma 17). We have a similar case if $\Delta \in P$. In all cases, Δ has a triangle face t between two tetrahedra in $M \cup P \cup S'$. Since $t \notin \partial O$, $c(\Delta) \cap c(\partial O) \neq c(\partial \Delta)$. According to Theorem 3 and since $\Delta \notin O$, $c(\Delta) \cap c(\partial O)$ is not 2D-pure or is a 2-ball, but it is not a 2-sphere.

Last we conclude. We use Theorem 11 using $A = M \cup S'$ and see that $c(\Delta) \cap c(\partial O)$ is not a 2-ball for every tetrahedron $\Delta \in P$. Now $c(\Delta) \cap c(\partial O)$ is not a 2-ball for every tetrahedron $\Delta \in P \cup S'$. Since $c(\Delta) \cap c(\partial O)$ is neither 2-ball nor 2-sphere, $c(\Delta) \cap c(\partial O)$ is not non-empty 2D-pure (Theorem 3). Since $\partial \Delta \cap \partial O \neq \emptyset$, $c(\Delta) \cap c(\partial O)$ is not 2D pure. □

7 Conclusion

This article overviews shelling properties for the surface reconstruction application. In this context, a shelling is a series of tetrahedron sets $O_0 \cdots O_n$ in a 3D Delaunay triangulation T, generated by a greedy algorithm that adds one tetrahedron to every O_i, such that all boundaries ∂O_i have the same topology in almost all cases.

If O_0 includes only one tetrahedron, all O_i are 3-balls and Combinatorial Topology works study the shellability of a 3-ball O: is there a shelling such that $O_n = O$? Several cases are possible for greedy shelling algorithms that try to generate O: always success (O is extendably shellable), sometimes success (O is shellable but non-extendably shellable), and always failure (O is non-shellable). Furthermore, there are 3 invariant numbers (the h-numbers) for all shellings of O.

We also show that there is a shelling started from O_0 and ended to O, if $O_0 \subset O \subseteq T$ and both O_0 and O are star-shaped with respect to a same point. This generalizes a known result of shellability (if $O = T$ and O_0 has a single tetrahedron) and we use this property to qualitatively explain why the greedy algorithm does not have excessive failures and provides most of the tetrahedra enclosed by the final surface.

Last we provide a family of non-shellable 3-balls and examples of visual artifacts that a shelling alone cannot remove. Such visual artifacts occur in previous surface reconstruction results and are due to an incomplete growing of O in the free-space. Actually non-shelling algorithms remove some of them, but future work is needed to remove all of them.

A Proofs for prerequisites of star-shape shelling

A.1 Proof of Lemma 6

First we show that there are points $\mathbf{x}' \in \mathring{\Delta}_x$ and $\mathbf{y}' \in \mathbf{cx}' \setminus |A|$. Since $|A|$ is a closed set in $\mathbb{R}^3$, there is $\epsilon > 0$ such that $B(\mathbf{y}, \epsilon) \cap |A| = \emptyset$. Let $\tau \in \mathbb{R}$ be such that $\mathbf{x} - \mathbf{c} = \tau(\mathbf{y} - \mathbf{c})$. We have $\tau > 1$ since $\mathbf{y}$ is between $\mathbf{x}$ and $\mathbf{c}$. If $\mathbf{x} \in |\partial \Delta_x|$, there is a point $\mathbf{x}'$ very close to $\mathbf{x}$ such that $\mathbf{x}' \in \mathring{\Delta}_x$ and $\mathbf{y}' = \mathbf{c} + (\mathbf{x}' - \mathbf{c})/\tau \in B(\mathbf{y}, \epsilon)$. If $\mathbf{x} \notin |\partial \Delta_x|$, we set $\mathbf{x}' = \mathbf{x}$ and $\mathbf{y}' = \mathbf{y}$.

Second we show that there are tetrahedron $\Delta_y \in T \setminus A$ and $\mathbf{x}'' \in \mathring{\Delta}_x$ such that $\mathbf{y}'' = \mathbf{c} + (\mathbf{x}'' - \mathbf{c})/\tau \in \mathring{\Delta}_y$. Since $|T|$ is convex and includes $\{\mathbf{c}, \mathbf{x}'\}$, $\mathbf{y}' \in |T|$. Since $\mathbf{y}' \notin |A|$, there is a tetrahedron $\Delta_y \in T \setminus A$ such that $\mathbf{y}' \in \Delta_y$. If $\mathbf{y}' \in |\partial \Delta_y|$, there is a point $\mathbf{y}''$ very close to $\mathbf{y}'$ such that $\mathbf{y}'' \in \mathring{\Delta}_y$ and $\mathbf{x}'' = \mathbf{c} + \tau(\mathbf{y}'' - \mathbf{c}) \in \mathring{\Delta}_x$.

Last we find $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. Since $\mathring{\Delta}_x$ and $\mathring{\Delta}_y$ are open, we define $B'_x = B(\mathbf{x}'', \tau\epsilon')$ and $B'_y = B(\mathbf{y}'', \epsilon')$ such that $B'_x \subset \mathring{\Delta}_x$ and $B'_y \subset \mathring{\Delta}_y$. Let V be the union for every edge $e \in c(T)$ of the plane(s) that include(s) both e and $\mathbf{c}$. Since we are in a non degenerate case, e and $\mathbf{c}$ are not collinear. Thus V is a finite union of planes and we cannot have $B'_x \subseteq V$. There is $\tilde{\mathbf{x}} \in B'_x \setminus V \subseteq \mathring{\Delta}_x$. This implies that $\mathbf{c}\tilde{\mathbf{x}}$ does not intersect the edges in $c(T)$ and $\tilde{\mathbf{y}} = \mathbf{c} + (\tilde{\mathbf{x}} - \mathbf{c})/\tau \in B'_y \subseteq \mathring{\Delta}_y$.

A.2 Proof of Lemma 7

First we show that the set of the triangles $t_i \in c(T)$ intersected by $(\mathbf{xy})$ are such that $\mathbf{x}_{\lambda_i} = (\mathbf{xy}) \cap t_i$ and the λ_i are distinct. If $\mathbf{x}_\lambda$ is in a triangle $t_\lambda \in c(T)$, $t_\lambda \cap (\mathbf{cx})$ is a line segment $\mathbf{x}_{\lambda'_0}\mathbf{x}_{\lambda'_1}$ (indeed the intersection of two closed convexes is a closed convex). Furthermore, $\lambda'_0 = \lambda'_1$ (otherwise, $(\mathbf{xy})$ is in the plane of t_λ and intersects an edge of t_λ) and we obtain $\mathbf{x}_\lambda = (\mathbf{xy}) \cap t_\lambda$. Let $\lambda_i \in \mathbb{R}$ such that $\mathbf{x}_{\lambda_i} \in t_i$. Now we show that $\lambda_i = \lambda_j$ implies $i = j$. Since $(\mathbf{xy})$ does not intersect the edges of t_i, $\mathbf{x}_{\lambda_i} \in \mathring{t}_i$. Thus $t_i = t_j$ (Lemma 5), i.e. $i = j$.

Second we note that there are at least two triangles t_i, i.e. $k \geq 1$. Indeed, $\mathbf{x} \in \mathring{\Delta}_x$ and $\Delta_x \in T$ imply that $(\mathbf{xy})$ intersects at least two triangles in $\partial \Delta_x \subset c(T)$.

Third the λ_i are ordered such that $\lambda_i < \lambda_{i+1}$ and we show that there is a tetrahedron $\Delta_i \in T$ such that $t_i \in \partial \Delta_i$, $t_{i+1} \in \partial \Delta_i$ and $\mathbf{x}_\lambda \in \Delta_i$ iff $\lambda \in [\lambda_i, \lambda_{i+1}]$. Let $\mathbf{m}_i = \mathbf{x}_{(\lambda_i + \lambda_{i+1})/2}$. Since $|T|$ is convex, there is a tetrahedron $\Delta_i \in T$ such that $\mathbf{m}_i \in \Delta_i$. Since $\mathbf{m}_i$ is not in a triangle of $c(T)$, $\mathbf{m}_i \in \mathring{\Delta}_i$. Therefore $\Delta_i \cap (\mathbf{xy})$ is a line segment $\mathbf{x}_{\lambda'_0}\mathbf{x}_{\lambda'_1}$ such that $\lambda'_0 \neq \lambda'_1$. This implies that $\mathbf{x}_\lambda \in \Delta_i$ iff $\lambda \in [\lambda'_0, \lambda'_1]$. Furthermore $\mathbf{x}_{\lambda'_\epsilon} \in |\partial \Delta_i|$ (indeed $\mathbf{x}_{\lambda'_\epsilon} \notin \mathring{\Delta}_i$ which is open). Thus there are distinct triangles t_j and t_l in $\partial \Delta_i$ such that $\mathbf{x}_{\lambda'_0} \in t_j$ and $\mathbf{x}_{\lambda'_1} \in t_l$ and $\lambda'_0 = \lambda_j$ and $\lambda'_1 = \lambda_l$. Since $\lambda_j < \lambda < \lambda_l$ implies $\mathbf{x}_\lambda \in \mathring{\Delta}_i$, which in turn implies $\mathbf{x}_\lambda \notin t_m$ (Lemma 5), we have $j + 1 = l$. Since $\mathbf{m}_i \in \mathbf{x}_{\lambda_j}\mathbf{x}_{\lambda_{j+1}}$, we have $j = i$. Thus $t_i \in \partial \Delta_i$, $t_{i+1} \in \partial \Delta_i$, $\lambda'_0 = \lambda_i$ and $\lambda'_1 = \lambda_{i+1}$.

Last we assume that a tetrahedron $\Delta \in T$ meets $(\mathbf{xy}) \cap \Delta \neq \emptyset$ and show that Δ is a Δ_i. The proof is similar as in the third step above. We have $(\mathbf{xy}) \cap \Delta = \mathbf{x}_{\tilde{\lambda}_0}\mathbf{x}_{\tilde{\lambda}_1}$ such that

$\tilde{\lambda}_0 \leq \tilde{\lambda}_1$ and $\mathbf{x}_{\tilde{\lambda}_\epsilon}$ is in a triangle $\tilde{t_\epsilon} \in \partial\Delta$. Since $(\mathbf{xy})$ does not intersect the Δ edges, $(\mathbf{xy})$ intersects $\mathring{\tilde{t_\epsilon}}$, which in turn implies that $\tilde{\lambda}_0 \neq \tilde{\lambda}_1$. Now there are distinct triangles t_j and t_l such that $\tilde{t_0} = t_j$, $\tilde{t_1} = t_l$, $\tilde{\lambda}_0 = \lambda_j$ and $\tilde{\lambda}_1 = \lambda_l$. Since $\tilde{\lambda}_0 < \lambda < \tilde{\lambda}_1$ implies $\mathbf{x}_\lambda \in \mathring{\Delta}$, which in turn implies $\mathbf{x}_\lambda \notin t_m$ (Lemma 5), we have $l = j+1$.

A.3 Proof of Theorem 7

Let $O \subseteq T$ be $\mathbf{c}$-star-shaped, a triangle $t \in \partial O$, a tetrahedron $\Delta \in O$ such that $t \in \partial\Delta$, and show that Δ and $\mathbf{c}$ are in the same t half-space. If $t \in \partial T$, both Δ and $\mathbf{c}$ are in the convex set T (indeed T is Delaunay) and thus there are in the same half-space of t. If $t \notin \partial T$, there is a tetrahedron $\Delta' \in T \setminus O$ such that $t = \Delta \cap \Delta'$. Let $\mathbf{b}$ be the barycentre of t. Since $\mathbf{c}$ is not in the t plane, $\mathring{\Delta} \cap \mathbf{bc} \neq \emptyset$ or $\mathring{\Delta}' \cap \mathbf{bc} \neq \emptyset$. Assume (reductio ad absurdum) that $\mathring{\Delta}' \cap \mathbf{bc} \neq \emptyset$. Since O is $\mathbf{c}$-star-shaped, $\mathbf{bc} \subseteq |O|$. Therefore there is $\Delta'' \in O$ such that $\mathring{\Delta}' \cap \Delta'' \neq \emptyset$. Lemma 5 implies that $\Delta' = \Delta'' \in O$, which contradicts $\Delta' \in T \setminus O$.

Assume that $O \subseteq T$ is not $\mathbf{c}$-star-shaped and show that there is a tetrahedron $\Delta \in O$ and a triangle $t \in \partial O \cap \partial\Delta$ such that Δ and $\mathbf{c}$ are not in the same t half-space. Since O is not $\mathbf{c}$-star-shaped, there is a point $\mathbf{x}' \in |O| \setminus \{\mathbf{c}\}$ such that $\mathbf{cx}'$ is not included in $|O|$. Thus there is a point $\mathbf{y}' \in \mathbf{cx}' \setminus |O|$. Now we use Lemma 6 with $A = O$. Let $\Delta_x \in O$ be a tetrahedron such that $\mathbf{x}' \in \Delta_x$. Thus there are tetrahedron $\Delta_y \in T \setminus O$ and points $\mathbf{x} \in \mathring{\Delta}_x$ and $\mathbf{y} \in \mathring{\Delta}_y \cap \mathbf{cx}$ such that $(\mathbf{cx})$ does not intersect the edges in $c(T)$. Then we obtain a tetrahedron series Δ_i covering $\mathbf{xy}$ as described by Lemma 7. There are j and m such that $\Delta_j = \Delta_x \in O$ and $\Delta_m = \Delta_y \in T \setminus O$. Since $\mathbf{x}_0 = \mathbf{x} \in \Delta_j$ and $\mathbf{x}_1 = \mathbf{y} \in \Delta_m$, $j \leq m$ and $\lambda_m \leq 1$. Therefore there is l such that $j \leq l < m$, $\Delta_l \in O$, $\Delta_{l+1} \in T \setminus O$ and $\lambda_{l+1} \leq \lambda_m$. Let $\Delta = \Delta_l$ and the triangle $t = \Delta_l \cap \Delta_{l+1}$. Note that $\mathbf{x}_\lambda \in \Delta$ iff $\lambda \in [\lambda_l, \lambda_{l+1}]$, $\mathbf{x}_\lambda \in t$ iff $\lambda = \lambda_{l+1}$, $\lambda_{l+1} \leq 1$, $\mathbf{c} = \mathbf{x}_\mu$ with $\mu > 1$ ($\mathbf{x}_1$ is between $\mathbf{x}_0$ and $\mathbf{c}$). Thus Δ and $\mathbf{c}$ are in different half-spaces of t.

B Proofs for star-shape shelling

B.1 Proof of Lemma 12

We change the coordinate basis such that $\mathbf{v}_1^T = (0\ 0\ 0)$, $\mathbf{v}_2^T = (1\ 0\ 0)$, $\mathbf{v}_3^T = (0\ 1\ 0)$, $\mathbf{v}_4^T = (0\ 0\ 1)$. Thus

$$\begin{aligned} H_1' &= \{(x_2,x_3,x_4)^T \in \mathbb{R}^3, x_2+x_3+x_4 \geq 1\} \\ H_2' &= \{(x_2,x_3,x_4)^T \in \mathbb{R}^3, x_2 \leq 0\} \\ H_3' &= \{(x_2,x_3,x_4)^T \in \mathbb{R}^3, x_3 \leq 0\} \\ H_4' &= \{(x_2,x_3,x_4)^T \in \mathbb{R}^3, x_4 \leq 0\}. \end{aligned} \quad (29)$$

Now we see that $(x_2,x_3,x_4) \in H_1' \cap H_2' \cap H_3' \cap H_4'$ is impossible.

Assume that $\mathbf{c} \in H_1' \cap H_2 \cap H_3 \cap H_4$. We have $\mathbf{v}_1 \in H_2 \cap H_3 \cap H_4$ (indeed $\Delta = H_1 \cap H_2 \cap H_3 \cap H_4$). Since $H_2 \cap H_3 \cap H_4$ is convex and includes both $\mathbf{c}$ and $\mathbf{v}_1$, $\mathbf{cv}_1 \subset H_2 \cap H_3 \cap H_4$. Let plane $\pi_1 = H_1 \cap H_1'$. Thus $\mathbf{cv}_1 \cap \pi_1 \subseteq H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_1' = \Delta \cap \pi_1 = t_1$. We obtain $\mathbf{cv}_1 \cap \pi_1 \subseteq \mathbf{cv}_1 \cap t_1$. Since $\mathbf{c}$ and $\mathbf{v}_1$ are in different t_1 half-spaces, $\mathbf{cv}_1 \cap \pi_1 \neq \emptyset$ and we obtain $\mathbf{cv}_1 \cap t_1 \neq \emptyset$.

Assume that $\mathbf{c} \in H_1' \cap H_2' \cap H_3 \cap H_4$. The tetrahedron $\Delta = \mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ is split in two tetrahedra $\Delta_1 = \mathbf{v}_1\tilde{\mathbf{v}}_2\mathbf{v}_3\mathbf{v}_4$ and $\Delta_2 = \tilde{\mathbf{v}}_1\mathbf{v}_2\mathbf{v}_3\mathbf{v}_4$ using $\tilde{\mathbf{v}}_2 = \tilde{\mathbf{v}}_1 = \mathbf{v} = (\mathbf{v}_1 + \mathbf{v}_2)/2$. Let $\tilde{H_1}$ and $\tilde{H_2}$ be the half-spaces of the triangle $\mathbf{vv}_3\mathbf{v}_4$ such that $\mathbf{v}_1 \in \tilde{H_1}$ and $\mathbf{v}_2 \in \tilde{H_2}$. Note that Δ_2 is like Δ except that the vertex $\mathbf{v}_1$ is replaced by $\tilde{\mathbf{v}}_1$ and the half-space H_2 is replaced by $\tilde{H_2}$: $\Delta_2 = H_1 \cap \tilde{H_2} \cap H_3 \cap H_4$ (t_1 does not change, the half-spaces H_3 and H_4 does not change since their boundary planes include $\tilde{\mathbf{v}}_1$, $\mathbf{v}_1$ and $\mathbf{v}_2$ that are collinear). Assume $\mathbf{c} \in \tilde{H_2}$ (the other case $\mathbf{c} \in \tilde{H_1}$ is similar). We have $\mathbf{c} \in H_1' \cap \tilde{H_2} \cap H_3 \cap H_4$ (indeed, $\mathbf{c} \in H_1' \cap H_3 \cap H_4$ and $\mathbf{c} \in \tilde{H_2}$). According to the second assertion of the lemma applied to Δ_2, $\mathbf{c}\tilde{\mathbf{v}}_1 \cap t_1 \neq \emptyset$. Thus $\mathbf{cv} \cap (t_1 \cup t_2) \neq \emptyset$.

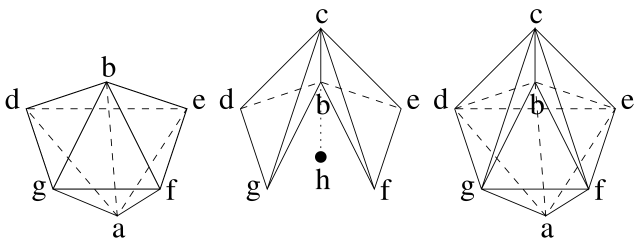

Fig. 18 A non-shellable star-shape O with respect to $\mathbf{h}$ such that $\mathbf{h} \notin |\partial O|$ (right): O is the union of $\mathbf{ab} \times \mathbf{d\text{-}e\text{-}f\text{-}g\text{-}d}$ (left) and $\{\mathbf{bcef}, \mathbf{bcgd}\}$ (middle).

C Non-shellable star-shapes in degenerate cases

Here we show that there are star-shapes O that are non shellable in degenerate cases. More precisely, we find $\mathbf{h}$-star-shapes O such that ∂O has a singular vertex and $\mathbf{h}$ is in a plane of a triangle in ∂O. Since ∂O is a 2-manifold if O is shellable (Theorem 4), we see that such an O is non-shellable.

For example, $O = \{\mathbf{habc}, \mathbf{hdef}\}$ is a $\mathbf{h}$-star-shape such that ∂O has a singular vertex $\mathbf{h}$. Fig. 18 shows another example O that is star-shaped with respect to a point $\mathbf{h} \notin |\partial O|$:

$$O = S_1 \cup \{\mathbf{bcef}, \mathbf{bcgd}\} \text{ where } S_1 = \mathbf{ab} \times \mathbf{d\text{-}e\text{-}f\text{-}g\text{-}d}. \quad (30)$$

Note that $\mathbf{h}$ is in the plane of the triangle $\mathbf{bcg} \in \partial O$ and $\mathbf{b}$ is a singular vertex of ∂O. Now we show that O can be star-shaped.

Property 2 Let $S_2 = \mathbf{bc} \times \mathbf{d\text{-}e\text{-}f\text{-}g\text{-}d}$ and $\mathbf{h} \in |S_1| \cap (\mathbf{bc})$. If both S_1 and $S_1 \cup S_2$ are convex, O is $\mathbf{h}$-star-shaped.

Proof Let $\mathbf{x} \in |O|$ and show that $\mathbf{xh} \subseteq |O|$. We have $\mathbf{x} \in |S_1|$ or $\mathbf{x} \in \mathbf{bcef}$ (or similarly $\mathbf{x} \in \mathbf{bcgd}$). Since S_1 is convex and $\mathbf{h} \in |S_1|$, $\mathbf{x} \in |S_1|$ implies $\mathbf{xh} \subseteq |S_1| \subseteq |O|$.

Now we assume $\mathbf{x} \in \mathbf{bcef}$ and show $\mathbf{xh} \cap \mathbf{bcde} = \mathbf{xh} \cap \mathbf{bce}$. Since $\mathbf{h} \in (\mathbf{bc})$ and $\mathbf{x} \in \mathbf{bcef}$, $\mathbf{hx}$ and $\mathbf{bcef}$ are in the same half-space of $\mathbf{bce}$. Since $\mathbf{bcde}$ is in the other half-space of $\mathbf{bce}$, $\mathbf{hx} \cap \mathbf{bcde}$ is included in the plane of $\mathbf{bce}$. We obtain $\mathbf{xh} \cap \mathbf{bcde} = \mathbf{xh} \cap \mathbf{bce}$ and similarly $\mathbf{xh} \cap \mathbf{bcfg} = \mathbf{xh} \cap \mathbf{bcf}$.

Since $S_1 \cup S_2$ is convex and $\mathbf{h} \in |S_1 \cup S_2|$, $\mathbf{xh} \subseteq |S_1 \cup S_2| = |O \cup \{\mathbf{bcde}, \mathbf{bcfg}\}|$. Thus

$$\mathbf{xh} = \mathbf{xh} \cap |O \cup \{\mathbf{bcde}, \mathbf{bcfg}\}| = \mathbf{xh} \cap (|O| \cup \mathbf{bce} \cup \mathbf{bcf}) \quad (31)$$

Since $\mathbf{bce} \subseteq |O|$ and $\mathbf{bcf} \subseteq |O|$, we obtain $\mathbf{xh} \subseteq |O|$. □

D Proof of Theorem 10

Lemma 18 *Assume that $A \subseteq T^\infty$ and $B \subseteq T \setminus A$. Then $c(A) \cap c(B)$ is a simplicial complex in $\mathbb{R}^3$. Furthermore, $c(A) \cap c(B) = c(\partial A) \cap c(\partial B)$.*

Proof This is Lemma 1 in [14] by replacing the set B by a single tetrahedron. The proof is very similar as the proof in Appendix A.1 of [14]. □

We show that every vertex $\mathbf{v} \in c(\partial(A \cup B))$ meets $(\partial(A \cup B))_{\mathbf{v}} = \mathbf{v} \times \mathbf{x}_*$ where $\mathbf{x}_*$ is a cycle. Thanks to Lemma 15, $\partial(A \cup B) = (\partial A \setminus \partial B) \cup (\partial B \setminus \partial A)$. Thus $\mathbf{v} \in c(\partial A)$ or $\mathbf{v} \in c(\partial B)$. We also have

$$(\partial(A \cup B))_{\mathbf{v}} = ((\partial A)_{\mathbf{v}} \setminus (\partial B)_{\mathbf{v}}) \cup ((\partial B)_{\mathbf{v}} \setminus (\partial A)_{\mathbf{v}}). \quad (32)$$

There are three cases. If $\mathbf{v} \in c(\partial A) \setminus c(\partial B)$, $(\partial B)_{\mathbf{v}} = \emptyset$. Thanks to Eq. 32, $(\partial(A \cup B))_{\mathbf{v}} = (\partial A)_{\mathbf{v}}$. Since ∂A is a 2-manifold, there is a cycle $\mathbf{x}_*$ such that $(\partial(A \cup B))_{\mathbf{v}} = (\partial A)_{\mathbf{v}} = \mathbf{v} \times \mathbf{x}_*$. The proof is the same if $\mathbf{v} \in c(\partial B) \setminus c(\partial A)$.

Now we consider the last case $\mathbf{v} \in c(\partial A) \cap c(\partial B)$. Since ∂A is a 2-manifold,

$$\exists \mathbf{a}_i \in c(\partial A), (\partial A)_{\mathbf{v}} = \mathbf{v} \times \mathbf{a}_*, \mathbf{a}_* = \mathbf{a}_1\text{-}\cdots\mathbf{a}_m\text{-}\mathbf{a}_1, \quad (33)$$

and $\mathbf{a}_*$ is a cycle. Since ∂B is a 2-manifold,

$$\exists \mathbf{b}_i \in c(\partial B), (\partial B)_{\mathbf{v}} = \mathbf{v} \times \mathbf{b}_*, \mathbf{b}_* = \mathbf{b}_1\text{-}\cdots\mathbf{b}_n\text{-}\mathbf{b}_1, \quad (34)$$

and $\mathbf{b}_*$ is a cycle. Since $c(\partial A) \cap c(\partial B) = c(A) \cap c(B)$ (Lemma 18) and $c(A) \cap c(B)$ is a 2-manifold with boundary, there are $\mathbf{c}_i \in c(\partial A) \cap c(\partial B)$ such that

$$(\partial A \cap \partial B)_{\mathbf{v}} = \mathbf{v} \times \mathbf{c}_*, \mathbf{c}_* = \mathbf{c}_1\text{-}\cdots\mathbf{c}_k, \quad (35)$$

and there are two cases: $\mathbf{c}_*$ is a cycle and $k \geq 3$, or all $\mathbf{c}_i$ are distinct and $k \geq 2$. Since $(\partial A \cap \partial B)_{\mathbf{v}} \subseteq (\partial A)_{\mathbf{v}}$ and $(\partial A \cap \partial B)_{\mathbf{v}} \subseteq (\partial B)_{\mathbf{v}}$, we have $\mathbf{c}_* \subseteq \mathbf{a}_*$ and $\mathbf{c}_* \subseteq \mathbf{b}_*$. Assume (reductio ad absurdum) that $\mathbf{c}_*$ is a cycle. Since $\mathbf{c}_*$ is included in $\mathbf{a}_*$ and $\mathbf{b}_*$ that are also cycles, $\mathbf{a}_* = \mathbf{c}_* = \mathbf{b}_*$. This implies $(\partial A)_{\mathbf{v}} = (\partial B)_{\mathbf{v}}$. Thanks to Eq. 32, $(\partial(A \cup B))_{\mathbf{v}} = \emptyset$. This contradicts $\mathbf{v} \in c(\partial(A \cup B))$.

Now the $\mathbf{c}_i$ are distinct and $k \geq 2$. Since the $\mathbf{c}_i$ are distinct and $\mathbf{c}_* \subseteq \mathbf{a}_*$, $k \leq m$. We shift the $\mathbf{a}_i$ such that $\mathbf{a}_i = \mathbf{c}_i$ if $1 \leq i \leq k$. Similarly $k \leq n$ and $\mathbf{b}_i = \mathbf{c}_i$ if $1 \leq i \leq k$. Thanks to Eq. 32, $(\partial(A \cup B))_{\mathbf{v}} = \mathbf{v} \times \mathbf{x}_*$ where

$$\mathbf{x}_* = \mathbf{a}_k\text{-}\cdots\mathbf{a}_m\text{-}\mathbf{a}_1 \cup \mathbf{b}_k\text{-}\cdots\mathbf{b}_n\text{-}\mathbf{b}_1. \quad (36)$$

Since $\mathbf{a}_1 = \mathbf{b}_1 = \mathbf{c}_1$ and $\mathbf{a}_k = \mathbf{b}_k = \mathbf{c}_k$,

$$\mathbf{x}_* = \mathbf{c}_k\text{-}\mathbf{a}_{k+1}\text{-}\mathbf{a}_{k+2}\text{-}\cdots\mathbf{a}_m\text{-}\mathbf{c}_1\text{-}\mathbf{b}_n\text{-}\mathbf{b}_{n-1}\text{-}\cdots\mathbf{b}_{k+1}\text{-}\mathbf{c}_k. \quad (37)$$

Then we show that $c(\mathbf{x}_*)$ has at least 3 distinct vertices. It has $\mathbf{c}_k$ and $\mathbf{c}_1$. Furthermore, $\mathbf{c}_1 \neq \mathbf{c}_k$ since $k \geq 2$. Assume (reduction ad absurdum) that it has no other vertex. Thus $m = k = n$. This implies $(\partial A)_{\mathbf{v}} = (\partial B)_{\mathbf{v}}$, which is impossible (see above).

Last we show that the vertices in the vertex sequence $\mathbf{x}_*$ are distinct. We have $\mathbf{c}_1 \neq \mathbf{c}_k$ since $k \geq 2$. We also have $\mathbf{c}_1 \neq \mathbf{a}_i$ if $i > k$, since $\mathbf{c}_1 = \mathbf{a}_1$ and the $\mathbf{a}_i$ are distinct (indeed, $\mathbf{a}_*$ is a cycle). Similarly, $\mathbf{c}_1 \neq \mathbf{b}_i$ if $i > k$, $\mathbf{c}_k \neq \mathbf{a}_i$ and $\mathbf{c}_k \neq \mathbf{b}_i$ if $i > k$. Assume (reductio ad absurdum) that $\mathbf{a}_i = \mathbf{b}_j$ if $i > k$ and $j > k$. Thus $\mathbf{va}_i \in c(\partial A) \cap c(\partial B)$. Since $c(\partial A) \cap c(\partial B)$ is 2D pure[2] (indeed, $c(\partial A) \cap c(\partial B)$ is a 2-manifold with boundary), there is a triangle $t \in (\partial A \cap \partial B)_{\mathbf{v}}$ such that $\mathbf{a}_i \in t$. This implies that $\mathbf{a}_i = \mathbf{a}_j$ with $i > k$ and $j \leq k$ (impossible).

[2] Addendum: use Lemma 24 in Appendix H.

E Proof of Theorem 12

First Theorem 17 details the splitting of a 2-sphere in $c(T)$ by a cycle. Then we show Theorem 12 (splitting of ∂S by the annulus N) by applying Theorem 17 to every cycle in ∂N.

Theorem 17 *Let $L \subseteq T$ be a 3-ball and $\mathbf{b}_* \in c(\partial L)$ be a cycle. There are B_i such that $B_1 \cup B_2 = \partial L$, $\partial B_1 = \partial B_2 = \mathbf{b}_*$, $c(B_1) \cap c(B_2) = c(\mathbf{b}_*)$, every B_i is strongly connected.*

Intuitively, the closed curve $\mathbf{b}_*$ splits the 2-sphere ∂L in two disjoint and connected sets (2-balls) of triangles B_1 and B_2 that have the common boundary $\mathbf{b}_*$. This is a discrete version of the Jordan Theorem for a 2-sphere encoded by a simplicial complex.

Proof Since the 3-ball $L \subseteq T$, ∂L is a 2-sphere (Theorem 5). Let $\mathbf{m} \in |\partial L|$ that is neither a vertex nor in an edge of $c(\partial L)$. Let φ be a homeomorphism such that $\varphi(|\partial L| \setminus \{\mathbf{m}\}) = \mathbb{R}^2$.

Let $G = (V, E)$ be the graph of the vertices and edges in $c(\partial L)$. Let $G^* = (V^*, E^*)$ be the adjacency graph of the triangles in ∂L, i.e. $V^* = \{v(t), t \in \partial L\}$ and $E^* = \{e(t_1, t_2), t_i \in \partial L, t_1 \cap t_2 \text{ is an edge }\}$. Both G and G^* have drawings (mappings) in $\mathbb{R}^2$ by φ. The drawing of G is $(\varphi(V), \varphi(E))$. The drawing of $v(t) \in V^*$ is the drawing of the barycentre of $t \in \partial L$. The drawing of $e(t_1, t_2) \in E^*$ is the drawing of a polygonal curve linking the barycentres of adjacent triangles t_1 and t_2 (the polygonal curve is included in $t_1 \cup t_2$ and intersects the edge $t_1 \cap t_2$). Let $F = \mathbf{b}_*$ and F^* be the dual edges of F, i.e. $F^* = \{e(t_1, t_2) \in E^*, t_1 \cap t_2 \in F\}$. Thanks to Proposition 4.6.1 in [6] (the plane duality theorem) and since G is connected and the drawings of G and G^* are dual and F is a cycle, F^* is a minimal cut of G^*.

Word "cut" means that the graph $(V^*, E^* \setminus F^*)$ is disconnected. Word "minimal" means that $(V^*, E^* \setminus F^*)$ becomes connected if we remove any edge from F^*. If we don't remove an edge from F^*, $(V^*, E^* \setminus F^*)$ has exactly two connected components $\{v(t), t \in B_1\}$ and $\{v(t), t \in B_2\}$: $B_1 \cup B_2 = \partial L$, $B_1 \cap B_2 = \emptyset$, and every B_i is non-empty strongly connected using the edges in $E^* \setminus F^*$.

Let edge $e \in \partial B_i$ and show $e \in \mathbf{b}_*$. Since ∂L is a 2-manifold and the B_i partition ∂L, there are triangles $t_i \in B_i$ such that $e = t_1 \cap t_2$. If $e(t_1, t_2) \in E^* \setminus F^*$, $v(t_1)$ and $v(t_2)$ are in the same connected component of $(V^*, E^* \setminus F^*)$ (impossible). Thus $e(t_1, t_2) \in F^*$ and $e \in F = \mathbf{b}_*$.

Let edge $e \in \mathbf{b}_*$ and show $e \in \partial B_i$. We have distinct triangles $t_i \in \partial L$ such that $e = t_1 \cap t_2$. Since $F = \mathbf{b}_*$, $e(t_1, t_2) \in F^*$. If $t_1 \in B_1$ and $t_2 \in B_1$ (reductio ad absurdum), we remove the edge $e(t_1, t_2)$ from F^* and add it to $E^* \setminus F^*$ without connecting B_1 and B_2, i.e. F is not a minimal cut (impossible). Thus $t_1 \in B_1$ and $t_2 \in B_2$; we obtain $e \in \partial B_i$.

Since $\partial B_i = \mathbf{b}_*$, $c(\mathbf{b}_*) \subseteq c(B_1) \cap c(B_2)$. Now we show $c(B_1) \cap c(B_2) \subseteq c(\mathbf{b}_*)$. Since $B_1 \cap B_2 = \emptyset$, there are not triangles in $c(B_1) \cap c(B_2)$. Let edge $e \in c(B_1) \cap c(B_2)$; e is included in a triangle in B_1 and another in B_2 and thus $e \in \mathbf{b}_*$. Let vertex $\mathbf{v} \in c(B_1) \cap c(B_2)$. Since ∂L is a 2-manifold, there is a triangle series $t_i \in (\partial L)_{\mathbf{v}}$ such that every $t_i \cap t_{i+1}$ is an edge. There is a $t_j \in B_1$ and $t_k \in B_2$. Therefore there is i such that $t_i \in B_1$ and $t_{i+1} \in B_2$. We see that the edge $e = t_i \cap t_{i+1}$ meets $e \in c(B_1) \cap c(B_2)$. Since $e \in \mathbf{b}_*$ (see above), $\mathbf{v} \in c(\mathbf{b}_*)$. □

Now we do the proof of Theorem 12.

First we introduce notations by applying Theorem 17 to cycles $\mathbf{b}_*$ and $\mathbf{c}_*$. There are B_i such that $B_1 \cup B_2 = \partial L$, $\partial B_1 = \partial B_2 = \mathbf{b}_*$, $c(B_1) \cap c(B_2) = c(\mathbf{b}_*)$ and every B_i is strongly connected. There are C_i such that $C_1 \cup C_2 = \partial L$,

$\partial C_1 = \partial C_2 = \mathbf{c}_*$, $c(C_1) \cap c(C_2) = c(\mathbf{c}_*)$ and every C_i is strongly connected.

Second we show that $N \subseteq B_2$ (similarly, $N \subseteq C_2$). Assume (reductio ad absurdum) that there are triangles $t_1 \in N \cap B_1$ and $t_2 \in N \cap B_2$. Since N is strongly connected and $N \subseteq B_1 \cup B_2$, we can choose t_1 and t_2 such that $t_1 \cap t_2$ is an edge. Therefore $t_1 \cap t_2 \in \partial B_i = \mathbf{b}_*$. However, an edge between two distinct triangles in N is $\mathbf{b}_i\mathbf{c}_i$ or $\mathbf{b}_{i+1}\mathbf{c}_i$ or $\mathbf{c}_{i+1}\mathbf{b}_i$ (contradiction). We obtain $N \subseteq B_2$.

Third we show that $c(N) \cap c(B_1) = c(\mathbf{b}_*)$ (similarly, $c(N) \cap c(C_1) = c(\mathbf{c}_*)$). Since $c(\mathbf{b}_*) \subseteq c(B_1)$ and $c(\mathbf{b}_*) \subseteq c(N) \subseteq c(B_2)$ and $c(B_1) \cap c(B_2) = c(\mathbf{b}_*)$, we have

$$c(\mathbf{b}_*) \subseteq c(N) \cap c(B_1) \subseteq c(B_1) \cap c(B_2) \subseteq c(\mathbf{b}_*). \tag{38}$$

Thus $c(N) \cap c(B_1) = c(\mathbf{b}_*)$.

Fourth we show that $B_1 \cap C_1 = \emptyset$. Since $\partial B_1 = \mathbf{b}_*$, there is a triangle $\tilde{t} = \mathbf{d}\mathbf{b}_0\mathbf{b}_1 \in B_1$. Assume (reductio ad absurdum) that $\tilde{t} \in C_1$. Thus $\mathbf{b}_0 \in c(C_1)$. Since $\mathbf{b}_0 \in c(N)$, we have $\mathbf{b}_0 \in c(C_1) \cap c(N) = c(\mathbf{c}_*)$, i.e. $\mathbf{b}_0$ is equal to a $\mathbf{c}_j$ (impossible). We see that every triangle $\tilde{t} = \mathbf{d}\mathbf{b}_0\mathbf{b}_1 \in B_1 \setminus C_1$. Assume (reductio ad absurdum) that there is a triangle $t \in C_1 \cap B_1$. Since B_1 is strongly connected and $\tilde{t} \in B_1 \setminus C_1$ and $t \in B_1 \cap C_1$, we can find triangles $t' \in B_1 \setminus C_1$ and $t'' \in B_1 \cap C_1$ such that $t' \cap t''$ is an edge. Therefore $t' \cap t'' \in \partial C_1 = \mathbf{c}_* \subseteq \partial N$. This implies that $t' \in N$ or $t'' \in N$, which is impossible since $B_1 \cap N \subseteq B_1 \cap B_2 = \emptyset$.

Fifth we show that $N \cup C_1 = B_2$. Since $B_1 \cap C_1 = \emptyset$ and $C_1 \subseteq B_1 \cup B_2$, $C_1 \subseteq B_2$. Thus $N \cup C_1 \subseteq B_2$ and we only need to show that $t''' \in B_2$ implies $t''' \in N \cup C_1$. Since B_2 is strongly connected and $\emptyset \neq N \cup C_1 \subseteq B_2$, there is a triangle series $t_i \in B_2$ such that $t_0 \in N \cup C_1$ and $t_n = t'''$ and every $t_i \cap t_{i+1}$ is an edge that is not in ∂B_2. Since $N \cap C_1 = \emptyset$,

$$\begin{aligned}\partial(N \cup C_1) &= (\partial N \setminus \partial C_1) \cup (\partial C_1 \setminus \partial N)\\ &= ((\mathbf{b}_* \cup \mathbf{c}_*) \setminus \mathbf{c}_*) \cup (\mathbf{c}_* \setminus (\mathbf{b}_* \cup \mathbf{c}_*))\\ &= \mathbf{b}_* = \partial B_2.\end{aligned} \tag{39}$$

Therefore every $t_i \cap t_{i+1}$ is not in $\partial(N \cup C_1)$. Now we have $t_{i+1} \in N \cup C_1$ if $t_i \in N \cup C_1$. Since $t_0 \in N \cup C_1$, we obtain $t''' = t_n \in N \cup C_1$.

Last we show $c(B_1) \cap c(C_1) = \emptyset$ and conclude using $B = B_1$ and $C = C_1$. Since $C_1 \subseteq B_2$ (see above), $c(B_1) \cap c(C_1) \subseteq c(B_1) \cap c(B_2) = c(\mathbf{b}_*)$. Furthermore $B_1 \subseteq C_2$ (indeed, $B_1 \cap C_1 = \emptyset$ and $B_1 \subseteq C_1 \cup C_2$ imply $B_1 \subseteq C_2$). Thus $c(B_1) \cap c(C_1) \subseteq c(C_2) \cap c(C_1) = c(\mathbf{c}_*)$. We obtain $c(B_1) \cap c(C_1) \subseteq c(\mathbf{b}_*) \cap c(\mathbf{c}_*) = \emptyset$.

F Proof of Theorem 14

We show that $\partial(A \cup B)$ is a connected 2-manifold that has the same genus as ∂A (therefore they are homeomorphic). The principle of the proof is the following. Theorem 10 implies that $\partial(A \cup B)$ is a 2-manifold. The set $D = \partial A \cap \partial B$ is a 2-ball[3]. Then we show that $\partial(A \cup B)$ is connected by studying the connectivity of $\partial A \setminus D$ and $\partial B \setminus D$ and since their closures include the 1-sphere ∂D. Last we use properties of Euler's characteristic $\chi(X)$ of $X \subseteq c(T)$ to show that both $\partial(A \cup B)$ and ∂A have the same genus. We remind that $\chi(X)$ is the number of the vertices plus number of the triangles minus the number of the edges in X, thus it is closely related to the genus of X (Sec. 2.6).

We start by the two following lemmas, which are useful to show that $\partial A \setminus D$ and $\partial B \setminus D$ are connected.

Lemma 19 *Let $M \subseteq c(T)$ be a 2-manifold*[4]*. Let $D \subset M$. Then $\partial(M \setminus D) = \partial D$ and $c(M \setminus D) \cap c(D) = c(\partial D)$.*

Proof First we show $\partial D = \partial(M \setminus D)$. Let $e \in c(M)$ be an edge. Since M is a 2-manifold, there are exactly two triangles t_1 and t_2 in M that includes e and $e = t_1 \cap t_2$. Since $D \subseteq M$, $e \in \partial D$ iff $t_1 \in D$ and $t_2 \in M \setminus D$ iff $e \in \partial(M \setminus D)$.

If a simplex $\sigma \in c(\partial D)$, $\sigma \in c(D)$ and $\sigma \in c(\partial(M \setminus D)) \subseteq c(M \setminus D)$. Therefore $\sigma \in c(M \setminus D) \cap c(D)$. Conversely, let $\sigma \in c(M \setminus D) \cap c(D)$ and show $\sigma \in c(\partial D)$. This means that there are triangles $t_1 \in D$ and $t_2 \in M \setminus D$ such that $\sigma \subseteq t_1 \cap t_2$. Assume that σ is an edge (case 1). Thus $\sigma = t_1 \cap t_2$. Since $D \subseteq M$ and M is a 2-manifold, the only triangles in M that includes σ are t_1 and t_2. Therefore there is only one triangle in D that includes σ and we obtain $\sigma \in c(\partial D)$. Assume that σ is a vertex (case 2). Since M is a 2-manifold, $M_{\mathbf{v}}$ is a triangle series $t'_i \in M$ such that every $t'_i \cap t'_{i+1}$ is an edge including $\mathbf{v}$[5]. There is j and k such that $t'_j = t_1 \in D$ and $t'_k = t_2 \in M \setminus D$. Thus we can find i such that $t'_i \in D$ and $t'_{i+1} \in M \setminus D$ (or vice versa). Now vertex σ is a face of the edge $e = t'_i \cap t'_{i+1}$ such that $e \in c(M \setminus D) \cap c(D)$ and we just showed (case 1) that $e \in c(\partial D)$. Therefore $\sigma \in c(\partial D)$. □

Lemma 20 *Assume that $M \subseteq c(T^\infty)$ is a connected 2-manifold*[6]*. If a 2-ball $D \subseteq M$, $M \setminus D$ is connected.*

Proof Let vertices $\mathbf{v}$ and $\mathbf{v}'$ in $c(M \setminus D)$. Since they are in $c(M)$ and M is connected, there is a path $\mathbf{v}_0$-$\mathbf{v}_1$-$\cdots\mathbf{v}_n \subseteq c(M)$ such that $\mathbf{v}_0 = \mathbf{v}$ and $\mathbf{v}_n = \mathbf{v}'$. If this path has an edge which is not in $c(M \setminus D)$, we modify it to obtain another path such that all its edges are in $c(M \setminus D)$ (thus $M \setminus D$ is connected).

Let j be the smallest index i such that $\mathbf{v}_i\mathbf{v}_{i+1} \notin c(M \setminus D)$. We show that $\mathbf{v}_j \in c(\partial D)$. Since $\mathbf{v}_j\mathbf{v}_{j+1} \in c(M) = c(M \setminus D) \cup c(D)$, $\mathbf{v}_j\mathbf{v}_{j+1} \in c(D)$. If $j = 0$, we have $\mathbf{v}_j \in c(M \setminus D) \cap c(D) = c(\partial D)$ (Lemma 19). If $j > 0$, $\mathbf{v}_{j-1}\mathbf{v}_j \in c(M \setminus D)$ and we have $\mathbf{v}_j \in c(M \setminus D) \cap c(D) = c(\partial D)$.

Let k be the greatest index i such that $\mathbf{v}_i\mathbf{v}_{i+1} \notin c(M \setminus D)$. Similarly, $\mathbf{v}_{k+1} \in c(\partial D)$.

Since D is a 2-ball, ∂D is a cycle of edges $\mathbf{d}_i\mathbf{d}_{i+1}$. There are integers l and m such that the path $\mathbf{d}_l$-$\mathbf{d}_{l+1}$-$\cdots\mathbf{d}_m \subset \partial D$ meets $\mathbf{d}_l = \mathbf{v}_j$ and $\mathbf{d}_m = \mathbf{v}_{k+1}$. Thanks to Lemma 19, every edge $\mathbf{d}_i\mathbf{d}_{i+1} \in c(M \setminus D)$. Last we concatenate $\mathbf{v}_0$-$\cdots\mathbf{v}_j$ and $\mathbf{d}_l$-$\cdots\mathbf{d}_m$ and $\mathbf{v}_{k+1}$-$\cdots\mathbf{v}_n$ to obtain a path $\mathbf{v}_0$-$\cdots\mathbf{v}_n \subseteq c(M \setminus D)$ such that $\mathbf{v}_0 = \mathbf{v}$ and $\mathbf{v}_n = \mathbf{v}'$. □

Now we show Theorem 14.

First we show that $\partial(A \cup B)$ is a connected 2-manifold. Thanks to Theorem 10, $\partial(A \cup B)$ is a 2-manifold. Furthermore $\partial A \setminus D$ is connected and $\partial B \setminus D$ is connected (Lemma 20) and $\emptyset \neq c(\partial D) \subseteq c(\partial A \setminus D) \cap c(\partial B \setminus D)$ (thanks to Lemma 19). Thus $\partial(A \cup B) = (\partial A \setminus D) \cup (\partial B \setminus D)$ is connected.

Second we show that $c(D) \cap c(\partial(A \cup B)) = c(\partial D)$. Lemma 19 using $M \in \{\partial A, \partial B\}$ implies $c(D) \cap c(\partial A \setminus D) = c(\partial D) = c(D) \cap c(\partial B \setminus D)$. Therefore

$$c(D) \cap (c(\partial A \setminus D) \cup c(\partial B \setminus D)) = c(\partial D). \tag{40}$$

We obtain the result using Lemma 15.

Third we show that $\chi(c(\partial(A \cup B))) = \chi(c(\partial A))$. Thanks to Lemma 15, $D \cup \partial(A \cup B) = \partial A \cup \partial B$. Since $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$,

$$\chi(c(D)) + \chi(c(\partial(A \cup B))) - \chi(c(D) \cap c(\partial(A \cup B)))$$

[3] Addendum: more details in Appendix H.

[4] Addendum: we also assume that M is a set of triangles.

[5] Addendum: $\mathbf{v}$ is a redundant notation of σ.

[6] Addendum: we also assume that M is a set of triangles.

	1	2	3	4	5	6	7	8	9	10	11	12
x	1	0	-1	0	2	0	-2	0	1	0	-1	0
y	0	1	0	-1	0	2	0	-2	0	1	0	-1
z	2	1	2	1	3	0	3	0	0	3	0	3

Table 1 3D coordinates of the 12 vertices in V.

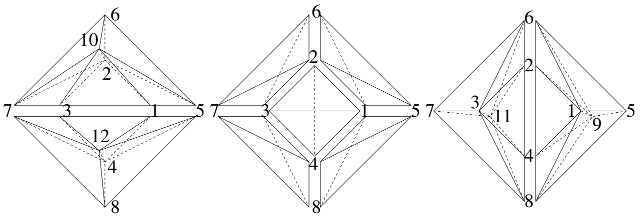

Fig. 19 Top views of the three subsets of Z. Left: the top subset is the union of the tetrahedra cycles $Z_{2.10}$ and $Z_{4.12}$. Middle: the center subset has five internal tetrahedra. Right: the bottom subset is the union of the tetrahedra cycles $Z_{1.9}$ and $Z_{3.11}$. Note that vertices are sometimes duplicated.

$$= \chi(c(\partial A)) + \chi(c(\partial B)) - \chi(c(\partial A) \cap c(\partial B)). \quad (41)$$

Furthermore[7], χ is known for 2-balls, cycles and 2-spheres:

$$\chi(c(D)) = \chi(c(\partial A) \cap c(\partial B)) = 1, \quad (42)$$

$$\chi(c(D) \cap c(\partial(A \cup B))) = \chi(c(\partial D)) = 0, \quad (43)$$

$$\chi(c(\partial B)) = 2. \quad (44)$$

Thus

$$1 + \chi(c(\partial(A \cup B))) - 0 = \chi(c(\partial A)) + 2 - 1. \quad (45)$$

Now we see that ∂A and $\partial(A \cup B)$ have the same genus.

G An example for Theorem 15

First we remind how Z is constructed in [24] using shortened notations: a vertex is an integer between 1 and 12, the tetrahedron with vertices $\{1, 2, 3, 4\}$ is written 1.2.3.4, the edge with vertices $\{1, 2\}$ is written 1.2, $1.2 \times$ 3-4-5-6 is the tetrahedron set $\{1.2.3.4, 1.2.4.5, 1.2.5.6\}$ etc. Let V be the set of the vertices $1, 2 \cdots 12$. The 3D coordinates of the vertices in V are given in Tab. 1. Then Z is defined as the disjoint union of three subsets of tetrahedra (Fig. 19). The center subset is

$$Z_0 = \{1.2.3.4, 1.2.5.6, 2.3.6.7, 3.4.7.8, 4.1.8.5\}. \quad (46)$$

The top subset is the union of the tetrahedron cycles $Z_{2.10}$ and $Z_{4.12}$:

$$Z_{2.10}{=}2.10 \times 5\text{-}6\text{-}7\text{-}3\text{-}1\text{-}5, Z_{4.12}{=}4.12 \times 7\text{-}8\text{-}5\text{-}1\text{-}3\text{-}7. \quad (47)$$

The bottom subset is the union of the tetrahedron cycles $Z_{1.9}$ and $Z_{3.11}$:

$$Z_{1.9}{=}1.9 \times 5\text{-}6\text{-}2\text{-}4\text{-}8\text{-}5, Z_{3.11}{=}3.11 \times 7\text{-}8\text{-}4\text{-}2\text{-}6\text{-}7. \quad (48)$$

We obtain

$$Z = Z_0 \cup Z_{1.9} \cup Z_{2.10} \cup Z_{3.11} \cup Z_{4.12}. \quad (49)$$

Second we check that Z is included in a set of tetrahedra T such that $c(T)$ is a 3D Delaunay triangulation.

[7] Addendum: since $c(A) \cap c(B)$ is a 2-ball (assumption of Theorem 14) and $c(A) \cap c(B) = c(\partial A) \cap c(\partial B)$ (thanks to Lemma 18, $c(\partial A) \cap c(\partial B)$ is a 2-ball.

Lemma 21 *Thanks to Tab. 1, Z is included in a 3D Delaunay triangulation whose vertex set is V.*

Proof Let $\mathbf{c}_\Delta$ and r_Δ be the center and radius of the circumscribing sphere of the tetrahedron Δ. Let

$$e = \min_{\Delta \in Z, \mathbf{u} \in V \setminus c(\Delta)} d^2(\mathbf{u}, \mathbf{c}_\Delta) - r^2_\Delta, \quad (50)$$

where d is the Euclidean distance between two 3D points. We have $e = 0.5 > 0$. Thus, for every $\Delta \in Z$, there is no $V \setminus c(\Delta)$ vertex in the (interior of the) sphere that includes the four Δ vertices. Therefore Z is included in a 3D Delaunay triangulation of V. □

Third we show that Z is the union[8] of four pipes and a 3-ball.

Lemma 22 *We have $Z = P \cup S$ where*

$$P = p(5.9.1, 6.2.10) \cup p(7.11.3, 6.2.10) \cup p(5.9.1, 8.4.12) \cup p(7.11.3, 8.4.12), \quad (51)$$

$$S = \{1.2.3.4, 1.9.2.4, 2.10.1.3, 3.11.2.4, 4.12.1.3\} \quad (52)$$

and S is a 3-ball[9].

Proof Eq. 15 is rewritten as

$$p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1) = \mathbf{b}_0\mathbf{c}_0 \times \mathbf{a}_0\text{-}\mathbf{a}_1\text{-}\mathbf{b}_1 \cup \{\mathbf{a}_0\mathbf{a}_1\mathbf{b}_1\mathbf{c}_0\} \cup \mathbf{b}_1\mathbf{c}_1 \times \mathbf{c}_0\text{-}\mathbf{a}_0\text{-}\mathbf{a}_1. \quad (53)$$

Thanks to Eq. 53,

$$p(5.9.1, 6.2.10) = 9.1 \times 5\text{-}6\text{-}2 \cup \{5.6.2.1\} \cup 2.10 \times 1\text{-}5\text{-}6$$
$$p(7.11.3, 6.2.10) = 11.3 \times 7\text{-}6\text{-}2 \cup \{7.6.2.3\} \cup 2.10 \times 3\text{-}7\text{-}6$$
$$p(5.9.1, 8.4.12) = 9.1 \times 5\text{-}8\text{-}4 \cup \{5.8.4.1\} \cup 4.12 \times 1\text{-}5\text{-}8$$
$$p(7.11.3, 8.4.12) = 11.3 \times 7\text{-}8\text{-}4 \cup \{7.8.4.3\} \cup 4.12 \times 3\text{-}7\text{-}8.$$

Now we see that

$$\begin{aligned} P = {} & (Z_0 \setminus \{1.2.3.4\}) \cup \\ & 9.1 \times 2\text{-}6\text{-}5\text{-}8\text{-}4 \cup 2.10 \times 1\text{-}5\text{-}6\text{-}7\text{-}3 \cup \\ & 11.3 \times 2\text{-}6\text{-}7\text{-}8\text{-}4 \cup 4.12 \times 1\text{-}5\text{-}8\text{-}7\text{-}3 \\ = {} & (Z_0 \setminus \{1.2.3.4\}) \cup (Z_{1.9} \setminus \{1.9.2.4\}) \cup \\ & (Z_{2.10} \setminus \{2.10.1.3\}) \cup (Z_{3.11} \setminus \{3.11.2.4\}) \cup \\ & (Z_{4.12} \setminus \{4.12.1.3\}) \\ = {} & Z \setminus S. \end{aligned} \quad (54)$$

and $S \subset Z$. Note that S is a 3-ball since it has a shelling started by the central tetrahedron 1.2.3.4 and the four other tetrahedra Δ_i are added to O_i such that $c(\Delta_i) \cap c(O_i) = c(t_i)$ where $t_i \in \partial 1.2.3.4$. □

Fourth, we show that $c(S) \cap c(P)$ is an annulus.

Lemma 23 *We have $c(S) \cap c(P) = c(N)$ where*

$$N = \{\mathbf{9}.\mathbf{2}.1, 2.\mathbf{1}.\mathbf{10}\} \cup \{3.\mathbf{2}.\mathbf{11}, \mathbf{10}.\mathbf{3}.2\} \cup \{\mathbf{11}.\mathbf{4}.3, 4.\mathbf{3}.\mathbf{12}\} \cup \{1.\mathbf{4}.\mathbf{9}, \mathbf{12}.\mathbf{1}.4\}. \quad (55)$$

The vertices in bold fonts form edges in ∂N:

$$\partial N = 9\text{-}2\text{-}11\text{-}4\text{-}9 \cup 1\text{-}10\text{-}3\text{-}12\text{-}1. \quad (56)$$

Proof Let $\sigma \in c(S) \cap c(P)$ be a simplex. Since vertices 5,6,7,8 are not in $c(S)$, σ does not have a vertex in $\{5, 6, 7, 8\}$. According[10] to Eq. 16, the only triangles in $\partial p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1)$ that does not have vertices $\mathbf{a}_0$ and $\mathbf{a}_1$ are $\{\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1\}$.

[8] Addendum: Z is the disjoint union.

[9] Addendum: we also have $S \cap P = \emptyset$ (thanks to Eq. 54).

[10] Addendum: thanks to Eq. 14, if $\sigma \in c(p(\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0, \mathbf{a}_1\mathbf{b}_1\mathbf{c}_1))$ has neither vertex $\mathbf{a}_0$ nor $\mathbf{a}_1$, $\sigma \in c(\mathbf{b}_0\mathbf{b}_1\mathbf{c}_0, \mathbf{b}_1\mathbf{c}_0\mathbf{c}_1)$.

Using Eq. 51 and $\mathbf{a}_0 \in \{5,7\}$ and $\mathbf{a}_1 \in \{6,8\}$, σ is included is[11] one of the triangles in $\tilde{N}$ where

$$\tilde{N} = \{9.2.1, 2.1.10\} \cup \{11.2.3, 2.3.10\} \cup \{9.4.1, 4.1.12\} \cup \{11.4.3, 4.3.12\}. \quad (57)$$

Therefore $c(S) \cap c(P) \subseteq c(\tilde{N})$. Conversely[12], $\tilde{N} \subseteq c(P)$ and $\tilde{N} \subseteq c(S)$ implies $c(\tilde{N}) \subseteq c(S) \cap c(P)$. Last we check that $N = \tilde{N}$. □

Thus $Z = P \cup S$ with $n = 4$ and $P = \cup_i P_i$[13] as in Theorem 15. Since Z is strongly non-shellable [24], $Z = P \cup S'$ and $S = S'$.

H Addendum

- Title "2-Manifold criterion for sets of triangles" is better than "2-Manifold criterion for simplicial complexes" for Sec. 2.5, since X in Sec. 2.5 is a set of triangles (X is not a simplicial complex).
- We need a basic Lemma two times in the paper:

  Lemma 24 *If a simplicial complex $X \subset c(T)$ is a 2-manifold with boundary, X is 2D pure.*

  Intuitively, if X is not 2D pure, there is a point in $|X|$ that has no 2D neighborhood in $|X|$. A proof is in Sec. H.1.
- Lemma 24 is used at the very end of Appendix D using $X = c(\partial A) \cap c(\partial B)$.
- Lemma 24 is used at the very beginning of Appendix F using $X = c(A) \cap c(B)$. Since this X is a 2-ball, X meets the Lemma's assumptions and $|X|$ is the union of the triangles in X. Since the set of these triangles is $D = \partial A \cap \partial B$, we see that D is a 2-ball.

H.1 Proof of Lemma 24

Assume (reductio ad absurdum) that X is not 2D pure. Thus there is a vertex that is not in an edge (both in X), or there is an edge that is not in a triangle (both in X). Since X is a 2-manifold with boundary, the vertex point is homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}^+$ in the first case (impossible). In the second case, we consider the middle point $\mathbf{m}$ of the edge and one of its small neighborhood N in $|X|$ that is included in the edge such that N is homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times \mathbb{R}^+$. Now we see that $N \setminus \{\mathbf{m}\}$ is homeomorphic to $\mathbb{R}^2$ minus a point or homeomorphic to $\mathbb{R} \times \mathbb{R}^+$ minus a point. This is impossible since the former is disconnected and the latters are connected.

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[11] Addendum: replace "is" by "in".

[12] Addendum: the four equations just below Eq. 53 imply $\tilde{N} \subseteq c(P)$, the S definition in Lemma 22 implies $\tilde{N} \subseteq c(S)$.

[13] Addendum: P_0, P_1, P_2 and P_3 are those in Eq. 51.