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Topology Preservation Within Digital Surfaces

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Abstract

Given two connected subsets $Y \subset X$ of the set of the surfels of a connected digital surface, we propose three equivalent ways to express that Y is homotopic to X. The first characterization is based on sequential deletion of simple surfels. This characterization enables us to define thinning algorithms within a digital Jordan surface. The second characterization is based on the Euler characteristics of sets of surfels. This characterization enables us, given two connected sets $Y \subset X$ of surfels, to decide whether Y is $n-$homotopic to X. The third characterization is based on the (digital) fundamental group.

Introduction

Digital surfaces of three dimensional objects have proved to be a fruitful model for visualization and analysis of the objects they represent ([1]), especially in the biomedical field. Efficient algorithms for extracting surfaces from a volume, and computing shape characteristics exist ([7]). Sometimes, the surface itself needs to be segmented since some particular points are defined on it. In 2D and 3D digital images, the skeleton of an object defined in the image is often used to analyze this object. Skeletons may be obtained by iterative deletion of simple points ([6]). From the very definition of a simple point, a skeleton thus constructed is *topologically equivalent* to the initial object.

In this paper, we first define a framework to study the "topology" of subsets of a digital surface. To do this we use the notion, already well known in digital topology, of a digital surface as defined for instance in [3], [11], [1] and [7], which is a surfel-based notion of a discrete surface. We introduce two adjacency relations, called $e-$adjacency and $v-$adjacency, analogous to the famous 2D notions of $4-$adjacency and $8-$adjacency (see [6] for example). It is of prime importance that all the notions considered in this paper be completely discrete, and as much as possible formally similar to the extensively used 2D notions considered for instance in [6]. However, as far as the notion of topology preservation is concerned, the non-planar framework of this paper leads us to several important changes with respect to the 2D framework.

We propose three equivalent characterizations for topology preservation in a digital surface. In other words, given Σ a connected digital surface and $Y \subset X$ two subsets of the set of the surfels of Σ, we propose three equivalent ways to express that Y is *homotopic* to X.

- The first characterization is based on sequential deletion of simple surfels (Definition 3 and Definition 4). This characterization enables us to define topology-preserving thinning algorithms within a digital Jordan surface.
- The second characterization is based on the Euler characteristics of sets of surfels (Definition 6 and Theorem 1). This characterization enables us, given two sets $Y \subset X$ of surfels, to decide by an efficient algorithm whether Y is $n-$homotopic to X or not.
- The third characterization is based on the (digital) fundamental group (Theorem 2). This invariant was initially introduced by T. Y. Kong, first in the 3D euclidean case ([4]), and then in a more general framework ([5]). The characterization of homotopy of sets through the fundamental group provides a solid foundation for our notion, since it shows that it is essentially similar to the corresponding notion of continuous topology.

An application is presented in Section 5: we have used our first characterization of topology preservation to skeletonize geometrical areas of the surface of a human brain in order to extract the *loci* cortical *sulci*.

1 Basic Definitions and Notations

1.1 Connectedness in Digital Spaces

Let Σ be a fixed set and let $X \subset \Sigma$. We denote by $\mathrm{card}(X)$ the number of elements of X and we denote $\overline{X} = \Sigma \setminus X$. In the following, we shall define an *adjacency relation* α on X to be an antireflexive symmetric binary relation on X. An *α-path* with a length p is a sequence $(x_0, \ldots, x_p)$ in which x_{i-1} is α-adjacent or equal to x_i for $i = 1, \ldots, p$. Such an α-path is called *closed* if and only if $x_0 = x_n$ and is called *simple* if the points x_i for $i \in \{0 \ldots p\}$ are pairwise distinct. Two elements x and y are said to be *α-connected* in X if there exists an α-path $(x_0, \ldots, x_p)$ in X with $x_0 = x$ and $x_p = y$. The $\alpha-$connectedness relation is an equivalence relation and we call *$\alpha-$connected components* its equivalence classes. We say that two sets are $\alpha-$adjacent if there exists two of their respective members which are $\alpha-$adjacent. A set is $\alpha-$adjacent to an element x if at least one element of the set is $\alpha-$adjacent to x. We denote by $\mathcal{C}_\alpha(X)$ the set of $\alpha-$connected components of X. We also define the $\alpha-$neighborhood $N_\alpha(x)$ of x by $N_\alpha(x) = \{y \in \Sigma \ / \ y \text{ is } \alpha - \text{adjacent to } x\}$.

1.2 Structure of a Digital Jordan Surface

Here we describe the model of a surface considered in this paper, and some local structures which can be defined on such a surface. Afterwards, we no longer consider the volume from which the surface is built, and we give some intrinsic definitions and results on subsets of a surface.

First we will recall some definitions, which can be found for example in [3] or [11], restricted to the three dimensional case. In the following discussion, voxels may be seen as unit cubes rather than points of $\mathbb{Z}^3$. We consider two kinds of adjacency between voxels. Two voxels are said to be $18-$*adjacent* if they share a face or an edge. They are said to be $6-$*adjacent* if they share a face. A *surfel* is a couple (c, d) of $6-$adjacent voxels. It can be seen as a unit square shared by c and d. A surface is a set of surfels.

Let $O \subset \mathbb{Z}^3$ be a 6−connected or 18−connected set. We consider Σ the set of all surfels of the form (x, y) with $x \in O$ and $y \in \overline{O}$: we call Σ the *surface of* O. It is possible to define an (already classical) adjacency relation, here called $e-$adjacency between surfels of Σ.

The definition of the $e-$adjacency relation depends on whether we consider O as 18−connected or 6−connected. Indeed, let us consider two voxels v and v' of O which are 18−adjacent but not 6−adjacent and without a common 6−neighbor in O (see Figure 1.a). There exists on the voxel v a surfel x which shares a fixed edge σ with three other surfels, two surfels of v' and one surfel of v. If we consider the object O as 6−connected, the surfel x is $e-$adjacent to the unique surfel of O which shares the edge σ with x and lies on the same surfel v of O as x. By contrast, if we consider the object O as 18−connected, the surfel x is $e-$adjacent to the unique surfel of the voxel v' which lies on the same voxel of $\overline{O}$ as x. Thus, for each considered type of connectivity 6, 18, an $e-$adjacency relation is well defined, and any surfel of Σ has exactly 4 neighbors for this relation.

The kind of surface thus defined satisfies the Jordan property ([3]): an $e-$connected surface separates the space into two parts, one of which is 6−connected, and the other one which is 18−connected. This kind of surface is widely used in image analysis and manipulation.

Now let s and s' be two $e-$adjacent surfels of Σ. The surfels s and s' share an edge. The pair $\{s, s'\}$ is called an *edgel*. It is important to distinguish an edgel from the underlying edge of cube. Indeed, some geometrical edges lead to two edgels, as the edge between the two vertices marked with filled circles on the object O made of two voxels represented on figure 1.a.

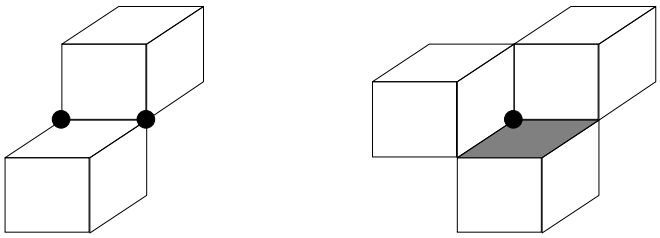
We define a *loop* in Σ as an $e-$connected component of the set of the surfels of Σ which share a vertex w. For example, in Figure 1.b, we see an object with three voxels. The vertex w marked with a filled circle defines two loops, one which can be seen on the figure and is composed of 3 surfels, and the other loop which is hidden and is composed of 6 voxels. Considering loops rather than vertices is a way to duplicate such vertices formally. As for edgels and edges, it is important to distinguish between loops and vertices.

We can define a unique cycle in any loop l: from a pair (s_1, s_2) of $e-$adjacent surfels in l choose s_3 the unique surfel of l which is $e-$adjacent to s_2 and which is distinct from s_1. By repeating this process we obtain a unique simple closed $e-$path of surfels which we call a *parametrization of the loop* l. The length of a loop ranges between 3 and 7. All the surfels of a loop share a common vertex. For this reason, we say that two surfels are $v-$*adjacent* (vertex adjacent) if they belong to a common loop.

Note that in the case of a planar surface orthogonal to one of the coordinate lines (i.e. in the case when the object O is a half space limited by a plane perpendicular to one of the coordinates axis), then the set of the surfels of the surface clearly identifies with the set of the pixels of $\mathbb{Z}^2$. Through this identification, $v-$adjacency corresponds to the classical 2D digital image notion of 8−adjacency and $e-$adjacency to the classical 4−adjacency.

Definition 1 (**$d-$cell**) *We associate a* dimension *to surfels, edgels, and loops, which is equal respectively to 2, 1, and 0. We can identify a surfel s with $\{s\}$. We call a surfel a 2-cell, an edgel a 1-cell, and a loop a 0-cell.*

If a surfel is a member of a loop or of an edgel, we also say that it is *incident* to this loop or this edgel. Moreover, whenever an edgel is a subset of a loop, we also say that it is incident to



(a) (b)

Figure 1: Counter example and example

this loop. We see that each surfel is incident to 4 loops and 4 edgels, and each edgel is incident to 2 loops.

Remark 1 We can consider a structure of a cellular complex for the surface. This even leads to a structure of a *combinatorial surface*, as defined in [2]. However, it is important that for this structure, a $0-$cell indeed corresponds to a loop, and not to a vertex, and a $1-$cell corresponds to an edgel, and not to an edge. We shall not use this notion of a cellular complex in the sequel, since in this framework it would make the formalism even heavier than it is by adding here a new set of definitions.

In the sequel, we make the assumption that each loop of the surface of our object O is a topological disk (i.e. has Euler $v-$characteristics equal to 1, see Definition 6). This is equivalent to assuming that any two $v-$adjacent surfels which are not $e-$adjacent cannot both belong simultaneously to two given distinct loops. For instance, in Figure 1, the object (a) does not satisfy this hypothesis: The two loops corresponding to the vertices marked with filled circles contain two non $e-$adjacent surfels in common. By contrast, the object (b) satisfies our hypothesis. We can express this assumption on the object O the surface of which we consider, saying that we assume that if O is considered as $18-$connected and x and y are two $18-$adjacent voxels of O which are not $6-$adjacent, one of the two following properties is satisfied:

1. The voxels x and y have an $18-$neighbor (or $6-$neighbor) in O in common;
2. The voxels x and y have two $26-$neighbors in O in common which are themselves $26-$adjacent.

We must make the same assumption on $\overline{O}$ if O is considered as $6-$connected.

In the sequel of this paper, we consider Σ a fixed $e-$connected component of a digital surface, and $n \in \{e, v\}$. We also denote by $\overline{n}$ the element of $\{e, v\}$ such that $\{\overline{n}, n\} = \{e, v\}$. We denote respectively by $\mathcal{L}_\Sigma$ and $\mathcal{E}_\Sigma$ the set of loops and the set of edgels of the surface Σ. We denote $\mathcal{L}_\Sigma(x) = \{l \in \mathcal{L}_\Sigma \ / \ x \in l\}$ and $\mathcal{E}_\Sigma(x) = \{\epsilon \in \mathcal{E}_\Sigma \ / \ x \in \epsilon\}$. In the sequel, we consider X a nonempty subset of Σ analyzed with the $n-$connectivity. We analyze $\overline{X}$ with the $\overline{n}-$connectivity.

2 Simple surfels, Euler characteristics

Let $x \in \Sigma$. As we have already said, we assume that any loop in Σ is a topological disk. However, the neighborhood $N_v(x) \cup \{x\}$ of the surfel x is not always a topological disk (see the $v-$neighborhood of the grey surfel of the surface of Figure 1.b for instance). If this is the case, we have to define a topology on $N_v(x) \cup \{x\}$ under which it is a topological disk. Let us consider two surfels y and y' in $N_v(x) \cup \{x\}$. We say that y and y' are e_x-*adjacent* [respectively v_x-*adjacent*] if they are $e-$adjacent [respectively $v-$adjacent] and are contained in a common loop which contains x. We denote by $G_e(x, X)$ [respectively $G_v(x, X)$] the graph whose vertices are the surfels of $N_v(x) \cap X$ and whose edges are pairs of e_x-adjacent [respectively v_x-adjacent] surfels of $N_v(x) \cap X$. We denote by $\mathcal{C}_n^x(G_n(x, X))$ the set of all connected components of $G_n(x, X)$ which are $n-$adjacent to x. Note that $\mathcal{C}_n^x(G_n(x, X))$ is a set of subsets of the set of all surfels of Σ and not a set of surfels.

Definition 2 *We call x an $n-$*isolated surfel *if $N_n(x) \cap X = \emptyset$ and an $n-$*interior surfel *if $N_{\overline{n}}(x) \cap \overline{X} = \emptyset$.*

Definition 3 (Simple surfel) *A surfel x is called $n-$*simple *in X if and only if the number* $\mathrm{card}(\mathcal{C}_n^x(G_n(x, X)))$ *of connected components of $G_n(x, X)$ which are $n-$adjacent to x is equal to 1, and if x is not interior to X.*

Remark 2 Note that, as in the 2D case, if $x \in X$ is non $n-$isolated, then x is $n-$simple if and only if the number $\mathrm{card}(\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X})))$ of connected components of $G_{\overline{n}}(x, \overline{X})$ which are $\overline{n}-$adjacent to x is equal to 1. Moreover, if x is neither $n-$interior nor $n-$isolated we have: $\mathrm{card}(\mathcal{C}_n^x(G_n(x, X))) = \mathrm{card}(\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X})))$.

The purpose of this paper is to study the following notion:

Definition 4 (homotopy) *Let $Y \subset X \subset \Sigma$. The set Y is said to be* (lower) $n-$homotopic *to X if and only if Y can be obtained from X by sequential deletion of $n-$simple surfels.*

This notion of homotopy is formally analogous to a 2D discrete notion which has been thoroughly studied (see [10]). It intuitively corresponds to a discrete notion of "deformation retract".

Now we propose a definition in this framework of a topological invariant for subsets of a digital surface: the Euler $n-$characteristics. We shall prove later that it is coherent with the definition of an $n-$simple surfel. It is also in accordance with the continuous analog of a digital surface, and with the dimension associated with cells.

Definition 5 (Elementary Euler $n-$characteristics of a cell) *For $d \in \{0, 1, 2\}$ and for c a $d-$cell, we define the* elementary Euler characteristic of c in X *as*

$$\chi_n^d(X, c) = (-1)^d.\mathrm{card}(\mathcal{C}_n(c \cap X))$$

Note that the only case in which $\chi_n^d(X, c)$ can be different from 0, 1 and -1 is when c is a loop and $n = e$. We denote $\chi_n^2(X) = \sum_{s \in \Sigma} \chi_n^2(X, s)$, $\chi_n^1(X) = \sum_{\epsilon \in \mathcal{E}_\Sigma} \chi_n^1(X, \epsilon)$ and $\chi_n^0(X) = \sum_{l \in \mathcal{L}_\Sigma} \chi_n^0(X, l)$.

Definition 6 *we define the* Euler $n-$characteristics of X, *and we denote by* $\chi_n(X)$ *the following quantity:*

$$\chi_n(X) = \chi_n^0(X) + \chi_n^1(X) + \chi_n^2(X) = \text{card}(X) + \chi_n^1(X) + \chi_n^2(X) \quad (1)$$

Remark 3 Note that 0 cells have a weighted contribution to the Euler characteristics (i.e. their elementary Euler $n-$characteristics is not always 1). This is the most important difference with the classical definition of the Euler characteristics.

Also note that the Euler $e-$characteristics is different from the Euler $v-$characteristics. This comes from the fact that, as usual in digital topology, the "topology" of the objects is not the same according to which adjacency relation we use to analyze them. For instance, if we consider a set X composed of two $v-$adjacent surfels which are not $e-$adjacent, then this object has one $v-$connected component which we want to consider as a disk. The Euler $v-$characteristics of this set X is 1. However, the same set X has two $e-$connected components which are both composed of one surfel, hence the set X, when analyzed with the $e-$adjacency relation, is made of two "disjoint" disks. That is why the Euler $e-$characteristics of this set X is 2.

Definition 7 (Contribution of a surfel to the Euler $n-$characteristics) *For* $X \subset \Sigma$, *we define the contribution of a surfel* $x \in X$ *to* $\chi_n(X)$ *as* $\Delta_n(X,x) = \chi_n(X) - \chi_n(X \setminus \{x\})$.

In the same way as $\chi_n(X)$, the number $\Delta_n(X,x)$ can be broken up into three terms which denote the contribution of x to the elementary Euler characteristic of the cells of different dimension which contain x:

$$\begin{array}{rcccccc} \Delta_n(X,x) & = & \Delta_n^2(X,x) & + & \Delta_n^1(X,x) & + & \Delta_n^0(X,x) \\ & = & 1 & + & \Delta_n^1(X,x) & + & \Delta_n^0(X,x) \end{array}$$

Where the second and the third terms can be expressed in the following way:

$$\Delta_n^1(X,x) = \sum_{\epsilon \in \mathcal{E}_\Sigma(x)} \Delta_n^1(X,\epsilon,x) \qquad \Delta_n^0(X,x) = \sum_{l \in \mathcal{L}_\Sigma(x)} \Delta_n^0(X,l,x)$$

with $\Delta_n^1(X,\epsilon,x) = \chi^1(X,\epsilon) - \chi^1(X \setminus \{x\},\epsilon)$ and $\Delta_n^0(X,l,x) = \chi_n^0(X,l) - \chi_n^0(X \setminus \{x\},l)$. The number $\Delta_n^1(X,\epsilon,x)$ [respectively $\Delta_n^0(X,l,x)$] is called the *contribution of* x *to the elementary Euler* $n-$*characteristics of* ϵ *in* X [respectively *of* l *in* X].

3 Another Characterization for Homotopy of Sets

The purpose of this section is to prove the following theorem which, together with Theorem 2, justifies our definition of simple points:

Theorem 1 *If* $Y \subset X \subset \Sigma$, *then the two following properties are equivalent:*

1. Y *is* $n-$*homotopic to* X.

2. $\chi_n(X) = \chi_n(Y)$ *and each* $\overline{n}-$*connected component of* $\overline{Y}$ *contains a surfel of* $\overline{X}$*.*

Before proving this theorem, we should provide several concepts and lemmas. Let $X \subset \Sigma$ and x be a surfel in X. The surfel x is incident to 4 loops which can be cyclically ordered. Due to our hypothesis that any loop is a topological disk, any surfel of $N_v(x)$ belongs either to a single loop of $\mathcal{L}_\Sigma(x)$ or to exactly two loops of $\mathcal{L}_\Sigma(x)$, and the second case occurs if and only if the surfel is $e-$adjacent to x. Finally, as we have already seen, each loop of $\mathcal{L}_\Sigma(x)$ is an oriented cycle for the $e-$adjacency relation. From these properties, we see that any surfel in $N_v(x)$ is e_x-adjacent to exactly two other surfels of $N_v(x)$. In other words, the graph $G_e(x, \Sigma)$ is a simple closed curve. Consequently, similarly to a loop, $N_v(x)$ also is an oriented cycle: there exists a closed e_x-path in $N_v(x)$ which contains each surfel of $N_v(x)$ exactly once. Such a closed e_x-path is called a *parametrization* of $N_v(x)$. Therefore, two non e_x-adjacent surfels of $N_v(x)$ split $N_v(x)$ into two intervals.

Definition 8 (Support of a connected subgraph of $G_n(x, X)$**)** *We can also define the* support *of a connected induced subgraph* C *of* $G_n(x, X)$*, denoted by* $\widetilde{C}$*, as the union of the intervals of* $N_v(x)$ *which have two* $n-$*adjacent surfels of* C *as their extremities and are contained in a loop.*

Remark 4 The support $\widetilde{C}$ of C is an interval of $N_v(x)$ or is equal to $N_v(x)$. The support of C can be different from C only in the case $n = v$.

Now, let C be a connected component of $G_n(x, X)$. We define a surfel s as being *interior to* C if $\{s, s_1, s_2\} \subset \widetilde{C}$, where s_1 and s_2 are the previous and next surfels of s in a parameterization of $N_v(x)$. With the same notations, s is in the *immediate interior of* C if $s_1 \notin \widetilde{C}$ or if $s_2 \notin \widetilde{C}$. A surfel $s' \in N_v(x) \setminus \widetilde{C}$ which is e_x-adjacent to $\widetilde{C}$ is said to be in the *immediate exterior of* C. We define the *extremities* of C as the pairs of e_x-adjacent surfels, one surfel of the pair lying in the immediate interior of C and the other surfel of the pair belonging to the immediate exterior of C. Note that two given successive surfels of $N_v(x)$ belong simultaneously to only one loop of $\mathcal{L}_\Sigma(x)$. As a consequence, we can say that an extremity $\{s, s'\}$ of C is in a given loop without ambiguity. If x is neither $n-$isolated nor $n-$interior, each element of $\mathcal{C}_n^x(G_n(x, X))$ or $\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X}))$ is a proper subset of $N_v(x)$, and has two extremities. Moreover, the cardinalities of these two last sets are equal.

In the following definition, we consider a loop l and its position with respect to connected components of $G_{\overline{n}}(x, \overline{X})$ which are $\overline{n}-$adjacent to x. We shall further link this classification with the contribution of x to the elementary Euler characteristic of l. Furthermore, we suppose that x is neither $n-$interior nor $n-$isolated, so that each connected component $C \in \mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X}))$ has two extremities.

Definition 9 (Classification of loops) *Let us consider* $x \in X$ *and* $l \in \mathcal{L}_\Sigma(x)$*.*

- *If for any* $C \in \mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X}))$ *we have* $C \cap l = \emptyset$ *we will say that the loop* l *is* $n-$interior *to* X *in* $N_v(x)$*.*
- *If there exists* $C \in \mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X}))$ *such that* $C \subset l$ *and* l *contains the two extremities of* C*, we say that* l contains an $\overline{n}-$connected component $C \in \mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x, \overline{X}))$*.*

- *If there exists* $C \in \mathcal{C}^x_{\overline{n}}(G_{\overline{n}}(x, \overline{X}))$ *such that* l *contains exactly one extremity of* C *we say that* l contains an extremity *of* $G_{\overline{n}}(x, \overline{X})$.

- *If* l *intersects any* $C \in \mathcal{C}^x_{\overline{n}}(G_{\overline{n}}(x, \overline{X}))$, *but contains no extremity of* C, *then we say that* l is tangent *to* $\overline{X}$ *in* $N_v(x)$.

These four cases are mutually exclusive and represent all possible configurations of loops. We are now going to analyze more precisely the contribution of x to the Euler characteristic of loops. We shall classify the configurations of a loop which is incident to $x \in X$ with respect to the two surfels of l which are $e-$adjacent to x. We shall also link each configuration with the classification of loops given above.

Lemma 1 *Let* l *be a loop of* $\mathcal{L}_\Sigma(x)$. *We consider* x_1 *and* x_2 *the two surfels of* l *which are* $e-$*adjacent to* x. *One of the three following cases occurs:*

1. *If* $x_1 \in X$ *and* $x_2 \in X$.

   (a) *If* x_1 *and* x_2 *are* $e-$*connected in* $l \cap (X \setminus \{x\})$ *then we have* $\mathrm{card}(C_n(l \cap X)) = \mathrm{card}(C_n(l \cap (X \setminus \{x\}))) = 1$, *thus* $\Delta^0_n(X, l, x) = 0$. *The loop* l *is* $n-$*interior to* X.

   (b) *If* x_1 *and* x_2 *are not* $e-$*connected in* $l \cap (X \setminus \{x\})$ *then* $l \cap X$ *has one less* $e-$*connected component than* $l \cap (X \setminus \{x\})$. *Notice that* x_1 *and* x_2 *are always* $v-$*connected in* $l \setminus \{x\}$. *Hence,*

      i. *if* $n = e$: $\Delta^0_e(X, l, x) = -1$ *and the loop contains an* $\overline{n}-$*connected component in* $\mathcal{C}^x_{\overline{n}}(G_{\overline{n}}(x, \overline{X}))$.

      ii. *if* $n = v$: $\Delta^0_v(X, l, x) = 0$ *and the loop is* $n-$*interior to* X.

2. *If* $x_1 \notin X$ *and* $x_2 \notin X$.

   (a) *If* $l \cap (X \setminus \{x\}) = \emptyset$ *then* $\Delta^0_n(X, l, x) = 1$ *and the loop tangent to* $\overline{X}$.

   (b) *If* $l \cap (X \setminus \{x\}) \neq \emptyset$ *then we have:*

      i. *if* $n = e$: $\Delta^0_e(X, l, x) = 1$ *and the loop is tangent to* $\overline{X}$.

      ii. *if* $n = v$: $\Delta^0_v(X, l, x) = 0$ *and the loop contains an extremity of* $G_{\overline{n}}(x, \overline{X})$.

3. *If* $x_1 \notin X$ *and* $x_2 \in X$ *or* $x_1 \in X$ *and* $x_2 \notin X$. *Then* $\Delta^0_n(X, l, x) = 0$. *The loop contains an extremity of* $G_{\overline{n}}(x, \overline{X})$.

Lemma 2 *The value of the contribution* $\Delta_n(X, x)$ *of* x *to the Euler* $n-$*characteristics of* X *is:*

- *equal to* 1 *if* x *is an* $n-$*isolated or an* $n-$*interior surfel of* X;

- *equal to* 0 *if* x *is a simple surfel of* X;

- *strictly negative otherwise.*

Proof: In any case, from X to $X \setminus \{x\}$, one surfel is removed: $\Delta_n^2(X,x) = 1$. We distinguish three cases:

First case: x is an $n-$isolated surfel in X. For any ϵ of the 4 edgels incident to x we have $\Delta_n^1(X,\epsilon,x) = -1$. Hence $\Delta_n^1(X,x) = -4$. Concerning any l of the 4 loops incident to x, if $n = e$, we are in Case 2a or Case 2(b)i of Lemma 1, and if $n = v$ we can only be in Case 2a. In both subcases we have $\Delta_n^0(X,l,x) = 1$, so that $\Delta_n^0(X,x) = 4$. Finally, $\Delta_n(X,x) = 1$.

Second case: x is an $n-$interior surfel. For any edgel ϵ incident to x we have $\Delta_n^1(X,\epsilon,x) = 0$, so that $\Delta_n^1(X,x) = 0$. For any $l \in \mathcal{L}_\Sigma(x)$ we are in one of the subcases of Case 1 of Lemma 1. If $n = v$ we can be in Case 1a or in Case 1(b)ii, and if $n = e$ we are in Case 1a. In both subcases we have : $\Delta_n^0(X,l,x) = 0$, so that $\Delta_n^0(X,x) = 0$. Finally, $\Delta_n(X,x) = 1$.

Third case: x is neither $n-$isolated nor $n-$interior. Let $k = \mathrm{card}(\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x,\overline{X})))$. Let us prove that $\Delta_n(X,x) = 1-k$. We consider $C \in \mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x,\overline{X}))$. We also consider the "size" of C as the number of loops it intersects, and we distinguish two subcases:

- If C intersects only one single loop l. This case may happen only whenever $\overline{n} = v$. Then l contains C and we are in Case 1(b)i of Lemma 1, so that $\Delta_n^0(X,l,x) = -1$. Moreover, no edgel is incident to both x and a surfel of C.

- If C intersects more than one loop. Let j be this number of loops intersecting C. These loops are consecutive and their union contains $j-1$ edgels which are incident to both x and C. For any ϵ of these edgels, we have $\Delta_n^1(X,\epsilon,x) = -1$. We see that among the j considered loops, exactly two loops contain exactly one extremity of C, and for l any of these two loops we must be in Case 2(b)ii or in Case 3, so that $\Delta_n^0(X,l,x) = 0$. Furthermore, for any l of the $j-2$ other loops, l is tangent to $\overline{X}$ in $N_v(x)$, hence we must be in Case 2a or in Case 2(b)i, so that $\Delta_n^0(X,l,x) = 1$. Therefore, the sum of the contributions of x to the elementary Euler characteristic of the cells with dimension 0 and 1 incident to both x and C is equal to -1, since the contribution of x to the elementary Euler $n-$characteristics of any of the $j-2$ loops which are tangent to $\overline{X}$ in $N_v(x)$ is equal to 1, the contribution of x to the elementary Euler $n-$characteristics of any of the $j-1$ edgels is equal to -1, and the contribution of x to the elementary Euler $n-$characteristics of the 2 loops containing an extremity of C is equal to 0.

In the two previous subcases, the sum of the contributions of x to the elementary Euler $n-$characteristics of cells of dimension 0 and 1 incident to both x and C is equal to -1. Moreover, if a loop intersects two elements C, C' of $\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x,X))$, this loop contains an extremity of $\widetilde{C}$ and of $\widetilde{C'}$ so that, as we noticed above, the contribution of x to the elementary Euler $n-$characteristics of this loop is equal to 0. Now let us also consider the loops and edgels which do not intersect any element of $\mathcal{C}_{\overline{n}}^x(G_{\overline{n}}(x,\overline{X}))$ (i.e. loops which are $n-$interior to X in $N_v(x)$): the contribution of x to their Euler $n-$characteristics is 0 (see Lemma 1 the cases 1a and 1(b)ii for the loops). Consequently, the sum of the contributions of x to the elementary Euler $n-$characteristics of all $0-$cells and $1-$cells incident to x is equal to $-k$. Now, taking into account the $2-$cell $\{x\}$, the contribution of x to the Euler characteristic is $1-k$. Consequently, it is 0 if the surfel is simple ($k = 1$), and strictly negative if $k > 1$. □

Remark 5 Lemma 2 establishes another local characterization of $n-$simple surfels.

Proof of Theorem 1: $1 \Rightarrow 2$: follows directly from Lemma 2.
$2 \Rightarrow 1$: Our proof is constructive: we give a method which, without using the definition of simple surfels, finds them in the border of X. If $Y \neq X$, let B be an $\overline{n}-$connected component of $\overline{Y}$ such that B contains a surfel $s \in X$. From Property 2, let y be a surfel of $B \cap \overline{X}$ and let π be an $\overline{n}-$path in B from s to y. The last surfel of π which is in X is a point of $X \cap \overline{Y}$ and is $\overline{n}-$adjacent to $\overline{X}$. Let us consider the set F of all surfels of $X \setminus Y$ which are $\overline{n}-$adjacent to $\overline{X}$. We have just shown that F was non empty. The $n-$adjacency graph naturally induces a distance on X, by means of the shortest $n-$path from a surfel to another. Let us choose x one of the surfels of F whose distance to Y in the $n-$adjacency graph of X is maximal.

We denote $X_1 = X \setminus \{x\}$. Because of the maximality of the distance from x to Y in the $n-$adjacency graph of X, the set X_1 is $n-$connected. The surfel x is obviously neither $n-$isolated (it is $n-$connected to Y) nor $n-$interior (it is $\overline{n}-$adjacent to $\overline{X}$). Moreover, the condition on the $\overline{n}-$connected components of $\overline{X}$ and $\overline{Y}$ still hold for $\overline{X_1}$ and $\overline{Y}$. At last, using Lemma 2 we obtain $\chi_n(X) \leq \chi_n(X_1)$.

We can iteratively apply this selection-deletion process producing X_{i+1} from X_i until $X_i = Y$. Then, we see from Lemma 2 that we have only deleted $n-$simple surfels at each step, since otherwise we would have $\chi_n(X) < \chi_n(Y)$, which contradicts the hypothesis: $\chi_n(X) = \chi_n(Y)$. □

Remark 6 Under the hypothesis and notations of Theorem 1, we assume that Property 1 is satisfied. Then the order in which we remove simple points from X is not important: if x is an $n-$simple point in X which does not belong to Y, then Y is $n-$homotopic to $X \setminus \{x\}$.

4 Fundamental Group and Topology Preservation

4.1 The Digital Fundamental Group

First, if α and β are two $n-$paths such that the extremity of α is equal to the origin of β, we denote by $\alpha * \beta$ the concatenation of the two $n-$paths α and β.

Now we need to introduce the *$n-$homotopy relation* between $n-$paths. Let us consider $X \subset \Sigma$. First we introduce the notion of an *elementary deformation*. Two closed $n-$paths π and π' in X having the same extremities are said to be *the same up to an elementary deformation (with fixed extremities) in X* if they are of the form $\pi = \pi_1 * \gamma * \pi_2$ and $\pi' = \pi_1 * \gamma' * \pi_2$, the $n-$paths γ and γ' having the same extremities and being both contained in a common loop. Now, the two $n-$paths π and π' are said to be *$n-$homotopic (with fixed extremities) in X* if there exists a finite sequence of $n-$paths $\pi = \pi_0, \ldots, \pi_m = \pi'$ such that for $i = 1, \ldots, m$ the $n-$paths π_{i-1} and π_i are the same up to an elementary deformation (with fixed extremities).

Let $B \in X$ be a fixed surfel called the *base surfel*. We denote by $A_B^n(X)$ the set of all closed $n-$paths $\pi = (x_0, \ldots, x_p)$ which are contained in X and such that $x_0 = x_p = B$. The $n-$homotopy relation is an equivalence relation on $A_B^n(X)$, and we denote by $\Pi_1^n(X)$ the set of equivalence classes of this equivalence relation. The concatenation of closed $n-$paths is compatible with the $n-$homotopy relation, hence it defines an operation on $\Pi_1^n(X)$, which associates the class of $\alpha * \beta$ to the class of α and the class of β. This operation provides $\Pi_1^n(X)$

with a group structure. We call this group the *$n-$fundamental group of X*. The $n-$fundamental group defined using a point B' as base point is isomorphic to the $n-$fundamental group defined using a point B as base point if X is $n-$connected.

Now we consider $Y \subset X \subset \Sigma$ and $B \in Y$ a base point. A closed $n-$path in Y is a particular case of a closed $n-$path in X. Furthermore, if two closed $n-$paths of Y are $n-$homotopic (with fixed extremities) in Y, they are $n-$homotopic (with fixed extremities) in X. These two properties enable us to define a canonical morphism $i_* : \Pi_1^n(Y) \longrightarrow \Pi_1^n(X)$, which we call the morphism induced by the inclusion map $i : Y \longrightarrow X$. To the class of a closed $n-$path $\alpha \in A_B^n(Y)$ in $\Pi_1^n(Y)$ the morphism i_* associates the class of the same $n-$path in $\Pi_1^n(X)$.

4.2 A third Characterization of Homotopy of Sets

Lemma 3 *Let $Y \subset \Sigma$, let $B \in X$ and $x \in X$ an $n-$simple point distinct from B. Then any $n-$path c of $A_B^n(X)$ is $n-$homotopic (with fixed extremities) to an $n-$path contained in $X \backslash \{x\}$.*

Proof: First, if $c = (x_0, \ldots, x_p)$ is an $n-$path in X such that $x_0 \neq x$ and $x_p \neq x$, we define an $n-$path $P(c)$ as follows: For any maximal sequence $\sigma = (x_k, \ldots, x_l)$ with $0 \leq k \leq l \leq p$ of points of c such that for $i = k, \ldots, l$ we have $x_i \neq x$, we define $c(\sigma) = \sigma$. For any maximal sequence $\sigma = (x_k, \ldots, x_l)$ with $1 \leq k \leq l < p$ of points of c such that for $i = k, \ldots, l$ we have $x_i = x$, we define $c(\sigma)$ as equal to the shortest $n-$path in $N_v(x) \cap X$ from x_{l-1} to x_{k+1}. Now, $P(c)$ is the concatenation of all $c(\sigma)$ for all maximal sequences $\sigma = (x_k, \ldots, x_l)$ of points of c such that either for $i = k, \ldots, l$ we have $x_i \neq x$ or for $i = k, \ldots, l$ we have $x_i = x$.
It is easily seen that c is $n-$homotopic (with fixed extremities) to $P(c)$. □

Lemma 4 *Let $X \subset \Sigma$ and x be an $n-$simple surfel in X. We consider two $n-$paths δ and δ' having the same extremities, δ and δ' both being contained in $N_v(x) \cap X$. Then δ and δ' are $n-$homotopic with fixed extremities in $X \backslash \{x\}$.*

Proof: We prove that $\delta = (x_0, \ldots, x_p)$ (similarly δ') is $n-$homotopic with fixed extremities to the shortest $n-$path in $N_v(x) \cap X$ between its extremities (note that this shortest $n-$path is unique). This is done by iteratively deleting in δ, through an elementary deformation, the point x_i each time x_{i+1} is $n-$adjacent or equal to x_{i-1} until we obtain the desired shortest $n-$path. □

Lemma 5 *Let $X \subset \Sigma$, let $B \in X$ and $x \in X$ an $n-$simple surfel which is distinct from B. Then if two closed $n-$paths π and π' of $A_B^n(X \backslash \{x\})$ are $n-$homotopic (with fixed extremities) in X, they are $n-$homotopic in $X \backslash \{x\}$.*

Proof: Let $P(\pi)$ and $P(\pi')$ be the $n-$paths as defined in the proof of Lemma 3. Now, it is sufficient to prove that if π and π' are the same up to an elementary deformation, the two $n-$paths $P(\pi)$ and $P(\pi')$ are $n-$homotopic. Hence we assume π and π' are of the form $\pi = \pi_1 * \gamma * \pi_2$ and $\pi' = \pi_1 * \gamma' * \pi_2$, the $n-$paths γ and γ' having the same extremities and both being contained in a common loop l. Without loss of generality, we assume that $x \in l$.

Note that the construction of $P(c)$ for $c = \pi_1$ makes sense if we agree to only considering the portion of π_1 from its origin to the last point of π_1 which is different from x. Of course, a

similar construction makes sense for π_2. Now, $P(\pi)$ is of the form $P(\pi) = P(\pi_1) * \delta * P(\pi_2)$, with δ contained in $N_v(x) \cap X$. Similarly, $P(\pi')$ is of the form $P(\pi') = P(\pi_1) * \delta' * P(\pi_2)$, with δ' contained in $N_v(x) \cap X$. From Lemma 4, the two $n-$paths δ and δ' are $n-$homotopic with fixed extremities. Hence $P(\pi)$ and $P(\pi')$ are also $n-$homotopic (with fixed extremities). □

Lemma 6 *Let $X \subset \Sigma$, and let $x \in X$ be an $n-$simple point of X. The morphism $i_* : \Pi_1^n(X \backslash \{x\}) \longrightarrow \Pi_1^n(X)$ induced by the inclusion map is a group isomorphism.*

Proof: Lemma 3 implies that i_* is onto, and Lemma 5 implies that i_* is one to one. □

Lemma 7 *Let $X \subset \Sigma$ be $n-$connected. Let $x \in X$, and let C be a connected component of $G_n(x, X)$ which is $n-$adjacent to x. Given $\pi = (x_0, \ldots, x_p)$ a closed $n-$path in X, we consider $\nu(\pi, x, C)$ the number of times there are in π successive points of the form $(x_i \in C, x_{i+1} = x)$ minus the number of times there are in π successive points of the form $(x_i = x, x_{i+1} \in C)$. Then the number $\nu(\pi)$ is invariant when π ranges within a given $n-$homotopy (with fixed extremities) class of closed $n-$paths of X.*

Proof: We consider two closed $n-$paths π and π' in X which are the same up to an elementary deformation, and we prove that $\nu(\pi, x, C) = \nu(\pi', x, C)$. Let us consider l the elementary cycle in which the deformation is effective. It is sufficient to treat the case when $x \in l$. Let $\pi = \gamma_1 * \varepsilon * \gamma_2$ and $\pi' = \gamma_1 * \varepsilon' * \gamma_2$ with ε and ε' both included in l and having the same extremities a and b. We distinguish three case:

First case : $\{a, b\} \subset C$ or $\{a, b\} \cap C = \emptyset$. In this case, the number of times ε enters into C (necessarily through x) is equal to the number of times it gets out of C (necessarily through x). Since we can make the same observation for ε', we have $\nu(\pi, x, C) = \nu(\pi', x, C)$.

Second case : $a \in C$ and $b \notin C$. In this case, the number of times ε enters into C (necessarily through x) is equal to the number of times it gets out of C (necessarily through x) minus one. Since we can make the same observation for ε', we have $\nu(\pi, x, C) = \nu(\pi', x, C)$.

Third case : $a \notin C$ and $b \in C$. In this case, the number of times ε (respectively ε') enters into C (necessarily through x) is equal to the number of times it gets out of C (necessarily through x) plus one. □

Lemma 8 *Let $Y \subset Z \subset X \subset \Sigma$ be $n-$connected sets such that $Z \backslash Y$ is equal to the intersection of X with an $\overline{n}$-connected component of $\overline{Y}$, and the morphism $i_* : \Pi_1^n(Y) \longrightarrow \Pi_1^n(X)$ induced by the inclusion map $i : Y \longrightarrow X$ is an isomorphism. We assume that there exists a surfel $x \in Z \backslash Y$ which is $\overline{n}-$adjacent to a surfel of $\overline{X}$.*
Then there exists a point of Z which is $n-$simple in X and which does not belong to Y.

Proof: We consider the point x of $Z \backslash Y$ $\overline{n}-$adjacent to $\overline{X}$ and whose distance to Y in the $n-$adjacency graph of Z is maximal among all surfels of Z which are $\overline{n}-$adjacent to $\overline{X}$. (indeed, for $y \in Z$, since Z is $n-$connected, the distance from y to Y in the $n-$adjacency graph of Z is well defined). Let us prove that x is $n-$simple in X.

We assume by contraposition that x is not $n-$simple in X. Since x is neither isolated nor interior to X, we consider two distinct connected components C_1 and C_2 in $G_n(x, X)$ which are $n-$adjacent to x. The connected components C_1 and C_2 are both $\overline{n}-$adjacent to $\overline{X} \cap N_v(x)$.

The set C_1 contains either a surfel of Y, or a surfel of Z which is $\overline{n}-$adjacent of $\overline{X}$, Therefore, by maximality of the distance from x to Y in the $n-$adjacency graph of Z, there exists in any case an $n-$path in Z which does not contain the point x from a point of C_1 (and similarly for C_2) to a point of Y. Moreover, we may assume that this $n-$path intersects C_1 only once at its extremity.

Using the existence of these two $n-$paths and the $n-$connectedness of Y, we can easily construct a closed $n-$path π such that the number $\nu(\pi, x, C_1)$ introduced in Lemma 7 is equal to 1. From Lemma 7, this closed $n-$path π cannot be $n-$homotopic to a closed $n-$path π' contained in Y. Indeed in this case we would have $\nu(\pi', x, C) = 0$ since π' could not contain the point x. This contradicts the fact that i_* is onto. □

Theorem 2 *Let $Y \subset X \subset \Sigma$ be $n-$connected sets. Then the two following properties are equivalent:*

1. *The set Y is $n-$homotopic to X.*
2. *The morphism $i_* : \Pi_1^n(Y) \longrightarrow \Pi_1^n(X)$ induced by the inclusion map $i : Y \longrightarrow X$ is an isomorphism and each $\overline{n}-$connected component of $\overline{Y}$ contains a point of $\overline{X}$.*

Proof: The fact that the first property implies the second one is a direct consequence of Lemma 6. We prove that the second property implies the first one.

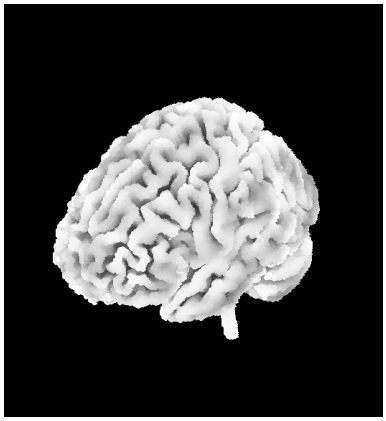
Let A be any non-empty intersection of X with an $\overline{n}-$connected component C of $\overline{Y}$. We set $Z = Y \cup A$. Since C contains a surfel y of $\overline{X}$, some surfel of $Z \backslash Y$ is $\overline{n}-$adjacent to $\overline{X}$. Indeed, the last surfel which belongs to Z of any $\overline{n}-$path in C from a point of $C \cap Z$ to y must be $\overline{n}-$adjacent to $\overline{X}$. Hence, from Lemma 8, the set Z contains an $n-$simple surfel x which does not belong to Y.

We set $X_1 = X \backslash \{x\}$. Then, since the morphism $i'_* : \Pi_1^n(X_1) \longrightarrow \Pi_1^n(X)$ induced by the inclusion map is an isomorphism, and so is the morphism i_*, The morphism $i''_* : \Pi_1^n(Y) \longrightarrow \Pi_1^n(X_1)$, which is equal to $(i'_*)^{-1} \circ i_*$, is also an isomorphism. Moreover, obviously any $\overline{n}-$connected component of $\overline{Y}$ must contain a point of $\overline{X_1}$. Hence we can prove again that if $X_1 \neq Y$, then X_1 contains an $n-$simple point which is not in Y. By induction, we obtain that Y is $n-$homotopic to X. □

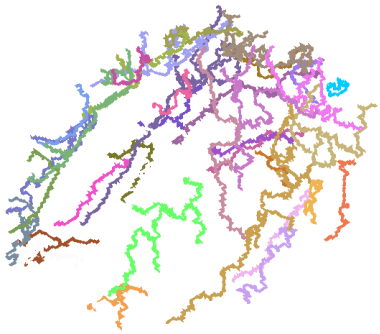
5 An Application to Topological Thinning

To extract the *loci* cortical *sulci* of a human brain, we first considered the surface of the brain as a classical 2D gray level image where the gray level is the mean curvature (Figure 2(a)), which was computed with a previously defined method ([7]). Then, we extracted the negative mean curvature part of the surface. Then, using our characterization of topology preservation, we obtained skeletons of these regions, and finally we filtered the result to remove insignificant branches (see Figure 2(b)).

To understand this method, let us consider the simpler artificial example of Figure 3. In this example, we have kept only the part of the surface with positive mean curvature.



(a) Mean curvature field on the surface of a brain (in light gray, the higher mean curvature surfels)



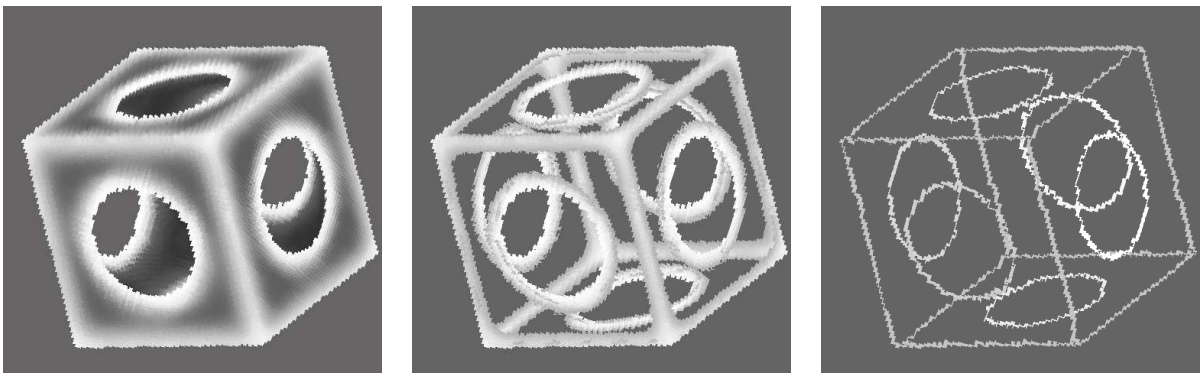
(b) After filtering the parts of the surface of negative mean curvature, we have extracted their center lines

Figure 2: Application of topological thinning.

The advantage of this method is that it uses the surface of an object as data. Such methods should to be developed since working on the surface of an object instead of on a volume saves memory and very often reduces the complexity of methods since the number of surfels of a surface is generally less than the number of voxels of a volume. Moreover, dealing with 2D objects, such as surfaces, is generally simpler than dealing with 3D objects.

Conclusion

We have provided three equivalent characterizations of topology preservation in a digital surface, and we have applied the notions thus obtained to topological thinning. We have made the assumption that loops are topological disks. This assumption is not too restrictive. In any case note that, in opposition to the characterization of $n-$simple surfels of Definition 3, the characterization of Lemma 2 seems to work also when we do not assume that loops are topological disks. It is probable that, using this general characterization of $n-$simple points in Definition 4 instead of that of Definition 3, we obtain a result similar to Theorem 1 without the assumption that loops are topological disks. Note that these results can be generalized to the case of the combinatorial surfaces defined in [2]. Several questions remain:
- The question of parallel thinning within a digital surface.
- In [8], we construct an explicit isomorphism from the fundamental group of any connected 2D object onto a free group. Furthermore, in [9] we give an algorithm to compute an algebraic presentation of the fundamental group of a subset of a digital surface. Finding efficient algorithms for the word problem and the isomorphism problem in this context is still an open question.



(a) Mean curvature field on the surface of the complement of a ball in a cube

(b) Subset of the surface composed of surfels with positive mean curvature

(c) After the thinning algorithm, the skeleton

Figure 3: An artificial example.

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